LARGE TIME WKB APPROXIMATION FOR MULTI-DIMENSIONAL SEMICLASSICAL SCHRÖDINGER-POISSON SYSTEM

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Abstract. We consider the semiclassical Schrödinger-Poisson system with a special initial data of WKB type such that the solution of the limiting hydrodynamical equation becomes time-global in dimensions at least three. We give an example of such initial data in the focusing case via the analysis of the compressible Euler-Poisson equations. This example is a large data with radial symmetry, and is beyond the reach of the previous results because the phase part decays too slowly. Extending previous results in this direction, we justify the WKB approximation of the solution with this data for an arbitrarily large interval of $\mathbb{R}_+$.

1. introduction

This paper is devoted to the study of the semiclassical limit $\varepsilon \to 0$ for the Cauchy problem of the semiclassical Schrödinger-Poisson system for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda V_{\mathrm{P}}^\varepsilon u^\varepsilon, \\ -\Delta V_{\mathrm{P}}^\varepsilon = |u^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \in L^\infty(\mathbb{R}^n), \quad V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x| \to \infty, \\ u^\varepsilon{}_{|t=0}(x) = A_0^\varepsilon(x) e^{i\frac{\Phi_0(x)}{\varepsilon}}, \end{cases} \tag{1.1}$$

where $n \geqslant 3$, ε is a positive parameter which corresponds to the scaled Planck constant, and λ is a real number. In addition, the "initial amplitude" A_0^ε is complex-valued and the "initial phase" Φ_0 is real-valued. Precise assumption on them is in Assumption 1.1. It is known that, if $n \geqslant 3$ then $V_{\mathrm{P}}^\varepsilon \in L^\infty(\mathbb{R}^n)$ is uniquely determined from $u^\varepsilon \in L^2 \cap L^\infty$ as

$$c_n(|x|^{n-2} * |u^\varepsilon|^2),$$

where c_n is a positive constant. Therefore, the Schrödinger-Poisson system (1.1) can be regarded as a special case of the Hartree equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda(|x|^\gamma * |u^\varepsilon|^2)u^\varepsilon. \tag{1.2}$$

For the well-posedness results on (1.1) and (1.2) for fixed $\varepsilon > 0$, see [8] and references therein.

In this paper, we are interested in the WKB type approximation for the solution of (1.1):

$$u^\varepsilon(t,x) \sim e^{i\frac{\phi(t,x)}{\varepsilon}}(\beta_0(t,x) + \varepsilon\beta_1(t,x) + \varepsilon^2\beta_2 + \cdots) \tag{1.3}$$

as $\varepsilon \to 0$. One way to justify (1.3) is to employ a modified Madelung transform

$$u^\varepsilon(t,x) = a^\varepsilon(t,x)\exp\left(i\frac{\phi^\varepsilon(t,x)}{\varepsilon}\right) \tag{1.4}$$

and consider the system

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon(0,x) = A_0^\varepsilon(x); \\ \partial_t\phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda V_{\mathrm{P}}^\varepsilon = 0, & \phi^\varepsilon(0,x) = \Phi_0(x); \\ -\Delta V_{\mathrm{P}}^\varepsilon = |a^\varepsilon|^2, \quad V_{\mathrm{P}}^\varepsilon \in L^\infty(\mathbb{R}^n), & V_{\mathrm{P}}^\varepsilon \to 0 \text{ as } |x|\to\infty. \end{cases} \tag{1.5}$$

Note that a^ε takes complex value. Our strategy is to obtain an expansion like

$$a^h = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots, \quad \phi^h = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots,$$

which yields (1.3) together with (1.4). This method is first applied to analytic data [15] and to Sobolev data [16] for certain class of defocusing nonlinearities, and is generalized to other local nonlinearities in [1, 4, 11] and to some nonlocal nonlinearities in [2, 7, 19, 20]. For this method, see also [6, 14]. One verifies that the principal part (a_0,ϕ_0) of $(a^\varepsilon,\phi^\varepsilon)$ solves, at least formally,

$$\begin{cases} \partial_t a_0 + \nabla\phi_0\cdot\nabla a_0 + \dfrac{1}{2}a_0\Delta\phi_0 = 0, & a_{0|t=0} = A_0, \\ \partial_t\phi_0 + \dfrac{1}{2}|\nabla\phi_0|^2 + \lambda V_{\mathrm{P}} = 0, & \phi_{0|t=0} = \Phi_0, \\ -\Delta V_{\mathrm{P}} = |a_0|^2, \quad V_{\mathrm{P}} \in L^\infty(\mathbb{R}^n), & V_{\mathrm{P}} \to 0 \text{ as } |x|\to\infty, \end{cases} \tag{1.6}$$

where $A_0 := \lim_{\varepsilon\to 0} A_0^\varepsilon$. In general, the classical solution of (1.6) breaks down in finite time by a formation of singularity. The space-time set where the solution ceases to be smooth is called caustic. At the caustic, the WKB type approximation (1.3) also breaks down. The shape of the caustic set depends on the initial data of (1.6). The aim of this paper is to justify the large time WKB with a special initial data of WKB type which does not cause the caustic.

Whether the caustic phenomena occurs or not boils down to the problem of global existence of the classical solution to (1.6). By choosing $\rho := |a_0|^2$ and $v := \nabla\phi_0$, we find that (1.6) becomes the compressible Euler-Poisson equations (see (2.1), below). The classical solutions of the compressible Euler-Poisson equations are studied in [9, 10, 12, 21]. We see from [12] that there is an example of the initial data which does not cause the caustic, provided $n = 1$ and $\lambda > 0$ (repulsive case, or defocusing case). For such initial data, the large time WKB analysis of (1.1) is shown in [20]. It is pointed out in [21] that, for $n \geqslant 3$ and under certain conditions such as radial symmetry, such example exists if $\lambda < 0$ (attractive case, or focusing case). Our results are based on this respect.

1.1. **Main result.** We denote by $H^s(\mathbb{R}^n)$ the usual Sobolve space: $H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n); (1-\Delta)^{s/2}f \in L^2(\mathbb{R}^n)\}$. Let us write $H^s = H^s(\mathbb{R}^n)$, for short.

Assumption 1.1. *Suppose $n \geqslant 3$ and $\lambda < 0$ (focusing case). Let "expansion level" N be a positive integer. We suppose the following conditions with some $s > n/2 + 2N + 1$:*

(1) *The initial amplitude $A_0^\varepsilon \in H^{s+1}(\mathbb{R}^n)$ writes*
$$A_0^\varepsilon = A_0 + \sum_{j=1}^{N} \varepsilon^j A_j + O(\varepsilon^{N+1}) \quad \text{in } H^{s+1}(\mathbb{R}^n).$$
Namely, there exist $A_j \in H^{s+1}$ ($j = 1, \dots, N$) such that $\limsup_{\varepsilon\to 0} \|A_0^\varepsilon - \sum_{j=0}^{N} \varepsilon^j A_j\|_{H^{s+1}}/\varepsilon^{N+1} < \infty$.

(2) *The limit A_0 of the initial amplitude is radially symmetric, that is, $A_0(x) = A_0(|x|)$: $\mathbb{R}_+ \to \mathbb{C}$. Moreover, there exist $\kappa \geqslant \lceil s\rceil + 3 - n/2$ and $\delta \in (0, 1/4]$ such that $A_0 \in C^{\lceil s\rceil+3}((0,\infty))$ satisfies*
$$A_0(r) \neq 0, \quad \forall r > 0; \quad \limsup_{r\to 0} \frac{|A_0^{(j)}(r)|}{r^{\kappa-j}} < \infty \tag{1.7}$$
for all $j \in [0, \lceil s\rceil + 3]$, and that the following limits exist and are nonzero:
$$\lim_{r\to 0} \frac{A_0(r)}{r^\kappa}, \quad \lim_{r\to\infty} r^{\frac{n}{2}+\lceil s\rceil+3+\delta} A_0^{(\lceil s+3\rceil)}(r), \tag{1.8}$$
where $\lceil s\rceil$ denotes the minimum integer larger than or equal to s.

(3) *The initial phase Φ_0 is a radial function $\Phi_0(x) = \Phi_0(|x|)$: $\mathbb{R}_+ \to \mathbb{R}$ given from A_0 by the formula*
$$\Phi_0(r) = \int_0^r \sqrt{\frac{2|\lambda|}{(n-2)s^{n-2}} \int_0^s |A_0(\sigma)|^2 \sigma^{n-1} d\sigma} ds + \text{const.} \tag{1.9}$$

We now state our main result of this paper.

Theorem 1.2. *Let Assumptions 1.1 be satisfied. Then, for any $T > 0$ and $\varepsilon > 0$ with $\varepsilon \leqslant C_1 e^{-C_2 T}$, there exist a solution $u^\varepsilon \in C([0,T]; H^{s+1})$ of (1.1), and there also exist $\phi_0 \in C([0,\infty); C^{2N+6})$ and $\beta_j \in C([0,\infty); H^{s-2j+3})$ such that the approximation*
$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{N-1}\beta_{N-1} + O(\varepsilon^N)) \quad \text{in } L^\infty([0,T]; H^{s-2N+1}) \tag{1.10}$$
holds, where C_1 and C_2 are positive constants. Moreover, C_2 is independent of N.

In other words, this theorem tells us that, for any $0 < \varepsilon \ll 1$, the solution u^ε of (1.1) exists and (1.10) is valid for $t \leqslant C_2' \log \frac{1}{\varepsilon} + C_1'$.

Remark 1.3. We remark that the initial amplitude A_0^ε is not necessarily a radial function, though its limit A_0 is supposed to be radially symmetric. A simple example of A_0 which satisfies Assumption 1.1 is $A_0(r) = r^\kappa\psi(r) + r^{-\frac{n}{2}-\delta}(1-\psi(r))$, where $\psi(r) \in C^\infty((0,\infty))$ is a function such that $0 \leqslant \psi \leqslant 1$, $\psi(r) = 1$ for $r < 1$, and $\psi(r) = 0$ for $r > 2$.

Remark 1.4. The initial data which satisfies Assumption 1.1 is not necessarily a small data. Indeed, if some $(A_0.\Phi_0)$ satisfies this assumption and if α is a complex number, then $(\alpha A_0, |\alpha|\Phi_0)$ also satisfies this assumption.

Remark 1.5. Notice that Φ_0 given by (1.9) belongs to $L^\infty(\mathbb{R}^n)$ if and only if $n \geqslant 5$. One also sees that $\nabla\Phi_0$ belongs to $L^p(\mathbb{R}^n)$ only if $p > 2^*$. This is due to the lack of decay at spatial infinity.

Theorem 1.2 follows from Theorems 1.6 and 1.10, below. To state them, we make some definitions and notation. For $n \geqslant 3$, $s > n/2+1$, $p \in [1,\infty]$, and $q \in [1,\infty]$, we define a function space $Y^s_{p,q}(\mathbb{R}^n)$ by

$$Y^s_{p,q}(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{Y^s_{p,q}(\mathbb{R}^n)}} \tag{1.11}$$

with norm

$$\|\cdot\|_{Y^s_{p,q}(\mathbb{R}^n)} := \|\cdot\|_{L^p(\mathbb{R}^n)} + \|\nabla\cdot\|_{L^q(\mathbb{R}^n)} + \left\|\nabla^2\cdot\right\|_{H^{s-2}(\mathbb{R}^n)}. \tag{1.12}$$

We denote $Y^s_{p,q} = Y^s_{p,q}(\mathbb{R}^n)$, for short. This space $Y^s_{p,q}$, introduced in [22], is a modification of the Zhidkov space X^s, which is defined, for $s > n/2$, by $X^s(\mathbb{R}^n) := \{f \in L^\infty(\mathbb{R}^n) | \nabla f \in H^{s-1}(\mathbb{R}^n)\}$. The Zhidkov space was introduced in [26] (see, also [13]). Roughly speaking, the exponents p and q in $Y^s_{p,q}$ indicate the decay rates at spatial infinity of the function and of its first derivative, respectively. Moreover, $X^s \risingdotseq Y^s_{\infty,2}$ if $n \geqslant 3$ and $Y^s_{2,2} = H^s$. We use the following notation:

$$Y^s_{I_1,q} := \cap_{p'\in I_1} Y^s_{p',q}, \qquad Y^s_{p,I_2} := \cap_{q'\in I_2} Y^s_{p,q'},$$

where I_1 and I_2 are intervals of $[1,\infty]$. These notation are sometimes used simultaneously. For example, $Y^s_{I_1,I_2} := \cap_{p'\in I_1, q'\in I_2} Y^s_{p',q'}$. For $q \in [1,n)$, we use the notation $q^* = nq/(n-q)$. The following two relations are sometimes useful: $q < q^* < \infty$; $q_1^* > q_2^*$ if and only if $q_1 > q_2$. If $f \in Y^s_{p,q}$ then $|\nabla f| \to 0$ as $|x| \to \infty$ by definition, and so $\|\nabla f\|_{L^\infty} \leqslant \left\|\nabla^2 f\right\|_{H^s} < \infty$ by Sobolev embedding, which means $Y^s_{p,q} = Y^s_{p,[q,\infty]}$. Similarly, it follows from the Sobolev embedding and Lemma 4.1, below, that

$$Y^s_{p,q} = Y^s_{p,[\min(q,2^*),\infty]}, \qquad Y^s_{p,q} \subset Y^s_{q^*,q} \quad \text{if } q < n.$$

We first claim that (1.6) has a radial global solution under Assumption 1.1. For a solution (a_0, ϕ_0) of (1.6), we here introduce

$$\eta(T) = \eta(T; s,p,q) = \sup_{t\in[0,T]} \left(\|a_0(t)\|_{H^{s+3}(\mathbb{R}^n)} + \|\nabla\phi_0(t)\|_{Y^{s+4}_{p,q}(\mathbb{R}^n)} \right), \tag{1.13}$$

which is the key value for combining the following two theorems.

Theorem 1.6. *Let Assumption 1.1 be satisfied, Then, there exists a radial global solution $(a_0,\phi_0)(t,x) = (a_0,\phi_0)(t,|x|)$ of (1.6) satisfying $a_0(t) \in H^{s+3}(\mathbb{R}^n)$, $\phi_0(t) \in C^{2N+6}$, and $\nabla\phi_0(t) \in Y^{s+4}_{(2^*,\infty],(2,\infty]}$ for all $t \geqslant 0$. Moreover, for any $p_0 > 2^*$ and $q_0 > 2+4\delta/n$, it holds that*

$$\|a_0(t)\|_{L^2(\mathbb{R}^n)} = \|A_0\|_{L^2(\mathbb{R}^n)}, \quad \|\nabla a_0(t)\|_{H^{s+2}(\mathbb{R}^n)} = o(1), \tag{1.14}$$

$$\|\nabla\phi_0(t)\|_{Y^{s+4}_{[p_0,\infty].[q_0,\infty]}(\mathbb{R}^n)} = o(1) \tag{1.15}$$

as $t \to \infty$. In particular, $\eta(T; s,p,q) = O(1)$ as $T \to \infty$ for all $p > 2^*$ and $q > 2+4\delta/n$.

Remark 1.7. Under the following three assumptions; (i) $n \geqslant 3$ and the radial symmetry; (ii) $A_0 \in L^2(\mathbb{R}^n)$; (iii) $\nabla\Phi(0) = 0$ (by the radial symmetry) and $\nabla\Phi_0$ decreases at spatial infinity; the classical solution (a_0, ϕ_0) of (1.6) is time-global if and only if $\lambda < 0$ and Φ_0 is given by (1.9) (see Theorem 2.2).

We now consider the WKB analysis of (1.1). Let us go back to the equation (1.5). It is proven in [2] that this system has a local solution $(a^\varepsilon, \phi^\varepsilon)$ for $0 \leqslant \varepsilon \ll 1$ and the solution can be expanded as

$$a^h = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots, \quad \phi^h = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$

for a class of initial data of WKB type (see, also [7]). However, the initial data satisfying Assumption 1.1 is out of framework of this results because the spatial decay of the phase function Φ_0 is slow. So, we extend the result in this direction (see Remark 1.11, below).

Assumption 1.8. *Let $n \geqslant 3$ and $\lambda \in \mathbb{R}$. Let N be a positive integer. We suppose the following conditions with some $s > n/2 + 2N + 1$:*
- *The initial amplitude A_0^ε satisfies (1) of Assumption 1.1.*
- *The initial phase $\Phi_0 \in C^{2N+4}(\mathbb{R}^n)$ satisfies $\nabla\Phi_0 \in Y^{s+2}_{p,q}(\mathbb{R}^n)$ for some $p \in (2^*, \infty]$ and $q \in (2, n)$ with $p \geqslant q$.*

Remark 1.9. Assumption 1.8 is weaker than Assumption 1.1. In particular, A_0 and Φ_0 are not necessarily radially symmetric.

Theorem 1.10. *Let Assumption 1.8 be satisfied. Suppose that (1.6) has a global solution (a_0, ϕ_0) which satisfies $\eta(T; s, p, q) < \infty$ for all $T < \infty$. Then, for any $T > 0$ and $\varepsilon > 0$ with $\varepsilon \leqslant C_1\eta(T)e^{-C_2\eta(T)T}$, there exits a solution $u^\varepsilon \in C([0, T]; H^{s+1})$, and there also exist $\beta_j \in C([0, \infty); H^{s-2j+1})$ such that the approximation*

(1.16)
$$u^\varepsilon = e^{i\frac{\phi_0}{\varepsilon}}(\beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^{N-1}\beta_{N-1} + O(\varepsilon^N)) \quad \text{in } L^\infty([0, T]; H^{s-2N+1})$$

holds, where C_1 and C_2 are positive constants. Moreover, C_2 depends only on n and s.

Remark 1.11. In [2] and [7], the assumption on the initial phase Φ_0 is that $\Phi_0 \in L^\infty$, $\nabla\Phi_0 \in L^{2^*} \cap L^{n-} \cap L^\infty$, and $\nabla^2\Phi_0 \in L^2 \cap L^\infty$ (up to a subquadratic polynomial). Here, we assume $\nabla\Phi_0 \in L^{2^*+} \cap L^\infty$ and $\nabla^2\Phi_0 \in L^{2+} \cap L^\infty$ at best. Remark that in our framework, Φ_0 does not necessarily belong to any Lebesgue space nor is not a polynomial, as in (1.9).

Remark 1.12. In Theorem 1.10, the constant C_2 is independent of the N and the initial data. This point is an improvement because this constant was proportional to N in [20]. On the other hand, C_1 depends on them. However, the influence of C_1 is much smaller than of C_2.

The rest of this paper is organized as follows: We prove Theorems 1.6 and 1.10 in Sections 3 and 5, respectively. Sections 2 and 4 are devoted to preliminary results for the proofs. In Appendix A, we prove some results on the radial compressible Euler-Poisson equations which we use for the proof of Theorem 1.6.

2. Preliminaries for the proof of Theorem 1.6

In this section, we collect some preliminary results which will be used for the proof of Theorem 1.6. In Subsections 2.1 and 2.2, we give an explicit example of the global solution to (1.6) based on results in [21]. We also show an elementary equality in Subsection 2.3.

2.1. **Global existence of the solution to the compressible Euler-Poisson system.** Let (a_0, ϕ_0) be a solution of (1.6). Then, one easily sees that $\rho := |a_0|^2$ and $v := \nabla\phi_0$ solve the compressible Euler-Poisson equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & \rho_{|t=0} = |A_0|^2, \\ \partial_t v + (v\cdot\nabla)v + \lambda\nabla V_{\mathrm{P}} = 0, & v_{|t=0} = \nabla\Phi_0, \\ -\Delta V_{\mathrm{P}} = \rho, \quad V_{\mathrm{P}} \in L^\infty(\mathbb{R}^n), & V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{2.1}$$

Now, we assume the radial symmetry and consider the radial version of (2.1)

$$\begin{cases} \rho_t + r^{-(n-1)}\partial_r(r^{n-1}\rho v) = 0, & \rho(0,r) = \rho_0(r) := |A_0(r)|^2; \\ v_t + v\partial_r v + \lambda\partial_r V_{\mathrm{P}} = 0, & v(0,r) = v_0(r) := \Phi_0'(r); \\ -r^{-(n-1)}\partial_r(r^{n-1}V_{\mathrm{P}}) = \rho,\ V_{\mathrm{P}} \in L^\infty, & V_{\mathrm{P}} \to 0 \text{ as } r \to \infty, \end{cases} \tag{2.2}$$

where unknowns are now real-valued functions $\rho : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $v : \mathbb{R}_+^2 \to \mathbb{R}$. Let us introduce several function spaces. For a nonnegative integer k, we define

$$D^k := \begin{cases} C([0,\infty)) & \text{if } k = 0, \\ C([0,\infty)) \cap C^k((0,\infty)) & \text{if } k \geqslant 1. \end{cases} \tag{2.3}$$

Similarly, we define

$$D^k_\rho := D^k \cap L^1((0,\infty), r^{n-1}dr), \qquad D^k_a := D^k \cap L^2((0,\infty), r^{n-1}dr)$$

for $k \geqslant 0$ and

$$D^k_\phi := \begin{cases} C^1([0,\infty)) & \text{if } k = 1, \\ C^1([0,\infty)) \cap C^k((0,\infty)) & \text{if } k > 1 \end{cases}$$

for $k \geqslant 1$. Let us start with the following theorem announced in [21]:

Theorem 2.1 (Corollary 1.17 in [21])**.** *Let $n \geqslant 3$, or $n \geqslant 1$ and $\lambda < 0$. Suppose $\rho_0 \in D^0_\rho$ is not identically zero and $v_0 \in D^1$ satisfies $v_0(0) = 0$ and $v_0 \to 0$ as $r \to \infty$. Then, the solution of (2.2) is global if and only if $n \geqslant 3$, $\lambda < 0$, and the initial data is of particular form*

$$v_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}}\int_0^r \rho_0(s)s^{n-1}ds}.$$

Suppose $\lambda < 0$ and $n \geqslant 3$. If $\rho_0 \in D^k_\rho$ for some $k \geqslant 0$ and if v_0 is as above, then $v_0 \in D^{k+1}$ and the corresponding solution is

$$\rho \in C^2([0,\infty), D^k_\rho) \cap C^\infty((0,\infty), D^k_\rho),$$
$$v \in C^1([0,\infty), D^{k+1}) \cap C^\infty((0,\infty), D^{k+1})$$

and given explicitly by

$$\rho(t,X(t,R)) = \rho_0(R)\left(1+\frac{nv_0(R)}{2R}t\right)^{-1}\left(1+\frac{2|\lambda|R\rho_0(R)}{(n-2)v_0(R)}t\right)^{-1},$$
$$v(t,X(t,R)) = v_0(R)\left(1+\frac{nv_0(R)}{2R}t\right)^{\frac{2}{n}-1},$$

where $X(t,R) = R(1+\frac{nv_0(R)}{2R}t)^{2/n}$. *Moreover, this solution is unique in* $C^2([0,\infty),D^0)\times C^1([0,\infty),D^1)$. *Furthermore, a pair of functions of* $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$ *defined as* $\mathbf{r}(t,x) := \rho(t,|x|)$ *and* $\mathbf{v}(t,x) := \frac{x}{|x|}v(t,|x|)$ *solve* (2.1) *in the distribution sense.*

In [21], the proof of this theorem was left incomplete. We illustrate the proof in Appendix A.

2.2. **Global existence of the solution to** (1.6)**.** We consider the radial version of (1.6):

(2.4)
$$\begin{cases} \partial_t a_0 + \partial_r\phi_0\partial_r a_0 + \dfrac{a_0}{2r^{n-1}}\partial_r(r^{n-1}\partial_r\phi_0) = 0, & a_{0|t=0} = A_0,\\ \partial_t\phi_0 + \dfrac{1}{2}(\partial_r\phi_0)^2 + \lambda V_{\mathrm{P}} = 0, & \phi_{0|t=0} = \Phi_0,\\ -\partial_r(r^{n-1}\partial_r V_{\mathrm{P}}) = r^{n-1}|a_0|^2, \quad V_{\mathrm{P}}\in L^\infty(\mathbb{R}_+), & V_{\mathrm{P}}\to 0 \text{ as } r\to\infty. \end{cases}$$

Theorem 2.2. *Suppose* $n\geqslant 3$, *or* $n\geqslant 1$ *and* $\lambda<0$. *Suppose* $A_0\in D_a^0$ *is not identically zero and* $\Phi_0\in D_\phi^2$ *satisfies* $\Phi_0'(0)=0$ *and* $\Phi_0'(r)\to 0$ *as* $r\to\infty$. *Then, the solution of* (2.4) *is global if and only if* $n\geqslant 3$, $\lambda<0$, *and the initial data is of particular form*

$$\Phi_0(r) = \int_0^r\sqrt{\frac{2|\lambda|}{(n-2)s^{n-2}}\int_0^s|A_0(\sigma)|^2\sigma^{n-1}d\sigma}\,ds + \text{const.} \tag{2.5}$$

Moreover, if $A_0\in D_a^k$ *for some* $k\geqslant 0$, *then the above* Φ_0 *belongs to* D_ϕ^{k+2} *and the corresponding global solution*

$$a_0\in C^2([0,\infty),D_a^k)\cap C^\infty((0,\infty),D_a^k)$$
$$\phi_0\in C^1([0,\infty),D_\phi^{k+2})\cap C^\infty((0,\infty),D_\phi^{k+2})$$

are given explicitly as

(2.6)
$$a_0(t,X(t,R)) = A_0(R)\left(1+\frac{nv_0(R)}{2R}t\right)^{-\frac{1}{2}}\left(1+\frac{2|\lambda|R|A_0(R)|^2}{(n-2)v_0(R)}t\right)^{-\frac{1}{2}},$$
$$\phi_0(t,X(t,R)) = \Phi_0(R) + \frac{t}{2}\left(\Phi_0'(R)^2 + \frac{n-2}{2}\int_0^R\frac{\Phi_0'(r)^2}{r}dr\right) + g(t),$$

where $X(t,R) = R(1 + \frac{n\Phi_0'(R)}{2R}t)^{2/n}$, *and* g *is a function of time given by*

$$g(t) = \begin{cases} \dfrac{2|\lambda|}{(n-2)(4-n)} \displaystyle\int_0^\infty \frac{|A_0(r)|^2 r^2}{\Phi_0'(r)} \left[\left(1 + \frac{n\Phi_0'(r)}{2r}t\right)^{\frac{4}{n}-1} - 1 \right] dr, & \text{if } n \neq 4, \\ |\lambda| \displaystyle\int_0^\infty \frac{|A_0(r)|^2 r^2}{\Phi_0'(r)} \log\left(1 + \frac{2\Phi_0'(r)}{r}t\right) dr, & \text{if } n = 4. \end{cases}$$

Furthermore, the solution is unique.

This theorem is an immediate consequence of Theorem 2.1 and the following lemma, which is a modification of [20, Lemma 3.1].

Lemma 2.3. *Let* $A_0 \in D_a^k$ *and* $\Phi_0 \in D_\phi^{k+2}$ *for some* $k \geqslant 0$*. Then, the following two statements are equivalent;*

(1) *the system* (2.4) *has a unique solution* (a_0, ϕ_0) *in* $C([0,T), D_a^k \times D_\phi^{k+2}) \cap C^1((0,T), D_a^k \times D_\phi^{k+2})$ *with* $(a_0,\phi_0)_{|t=0} = (A_0, \Phi_0)$*;*
(2) *the radial Euler-Poisson equations* (2.2) *has a unique solution* (ρ, v) *in* $C([0,T), D_\rho^k \times D^{k+1}) \cap C^1((0,T), D_\rho^k \times D^{k+1})$ *with* $(\rho, v)_{|t=0} = (|A_0|^2, \Phi_0')$.

Moreover, the maximal existence times of (a_0,ϕ_0) *and of* (ρ, v) *are the same.*

2.3. **An equality.** In the forthcoming section, we will investigate the regularity of the radial global solution given in Theorem 2.2. Especially, we investigate higher derivatives of a_0 and ϕ_0. The following equality is useful, which reflects the special structure of the initial data.

Lemma 2.4. *Let* $n \geqslant 3$ *and* $\lambda < 0$*. Suppose* $A_0 \in C^M((0,\infty))$ *for large integer* M*. Define*

$$v_0(r) = \sqrt{\frac{2|\lambda|}{(n-2)r^{n-2}} \int_0^r |A_0(s)|^2 s^{n-1} ds}.$$

Then, there exist real constants $\alpha_{j,k}$ *and* $\beta_{l,m,k}$ *such that the following equality holds for* $k \in [1, M+1]$*:*

$$\sum_{j=0}^k \alpha_{j,k} r^j v_0^{(j)} = \sum_{l=1}^k \sum_{m \in (\mathbb{N}\cup\{0\})^l, |m| \leqslant k-l} \beta_{l,m,k} \frac{\left(\prod_{i=1}^l \rho_0^{(m_i)}\right) r^{2l+|m|}}{v_0^{2l-1}}, \tag{2.7}$$

where $\rho_0 := |A_0|^2$, $g^{(m)}$ *denotes the* m*-th derivative of* g *with the convention* $g^{(0)} = g$*. Moreover,* $\alpha_{k,k} = 1 \neq 0$.

Proof. By definition, v_0 satisfies

$$\frac{n-2}{2} v_0 + r v_0' = \frac{|\lambda|}{n-2} \frac{\rho_0 r^2}{v_0}. \tag{2.8}$$

This implies that (2.7) holds if $k = 1$ and $a_{1,1} = 1$. Then, operating $r\partial_r$ to the both sides of (2.8) and substituting (2.8) in the right hand side, we obtain (2.7) with $k = 2$. Repeating this argument, we obtain the result. $\square$

3. Proof of Theorem 1.6

We are now in a position to prove Theorem 1.6. The global solution has been obtained in previous section. In Subsection 3.1, we check the regularity of the solution at the initial time $t = 0$. Then, we investigate the regularity of the solution for $t \geqslant 0$ and establish an estimate on the time-order as $t \to \infty$ of the norm of the solution in Subsection 3.2.

3.1. **Regularity at the initial time.** Let us first consider the regularity of the initial data of (1.6) which satisfies Assumption 1.1.

Proposition 3.1. *Let Assumption 1.1 be satisfied. Then,* $\mathbf{A}_0(x) = A_0(|x|)$ *and* $\mathbf{\Phi}_0(x) = \Phi_0(|x|)$ *satisfies* $\mathbf{A}_0 \in H^{s+3}(\mathbb{R}^n)$ *and* $\nabla\mathbf{\Phi}_0 \in Y^{s+4}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$, *respectively.*

Proof. **Step 1**. We first collect the decay property of A_0, $\rho_0 := |A_0|^2$, and $v_0 := \Phi_0'$. By (1.7) and (1.8) and the definition of v_0 , there exist positive constants c and C such that

$$\begin{aligned} cr^{\kappa} &\leqslant |A_0(r)| \leqslant Cr^{\kappa}, \\ cr^{2\kappa} &\leqslant \rho_0(r) \leqslant Cr^{2\kappa}, \\ cr^{\kappa+1} &\leqslant v_0(r) \leqslant Cr^{\kappa+1} \end{aligned} \tag{3.1}$$

as $r \to 0$. Then, we use (2.7) with $k = 1$ to obtain

$$v_0'(r) = \alpha\frac{v_0(r)}{r} + \beta\frac{|A_0(r)|^2 r}{v_0(r)} = O(r^{\kappa})$$

as $r \to 0$. Moreover, assumption (1.7) implies $r^j\rho_0^{(j)}(r) = O(r^{2\kappa})$ as $r \to 0$ for $j \in [1, \lceil s\rceil+3]$. By this estimate and (2.7) with $k = j$, we see by induction that

$$v_0^{(j)}(r) = O(r^{\kappa+1-j}) \tag{3.2}$$

as $r \to 0$ holds for $j \in [1, \lceil s\rceil + 4]$.

We next consider the decay rate as $r \to \infty$. Denote by L_∞ the second limit in (1.8). $L_\infty \neq 0$ by assumption. It follows by l'Hôpital's rule that

$$\lim_{r\to\infty} r^{\frac{n}{2}+j+\delta}A_0^{(j)}(r) = (-1)^{\lceil s\rceil-j}\frac{\Gamma(\frac{n}{2}+\lceil s\rceil+3+\delta)}{\Gamma(\frac{n}{2}+j+\delta)}L_\infty \neq 0$$

for $j \in [0, \lceil s\rceil + 3]$, where Γ is the Gamma function. Then, we see that $\lim_{r\to\infty} r^{n+j+2\delta}\rho_0^{(j)}(r)$ exists for $j \in [0, \lceil s\rceil + 3]$. Once they exist, then

$$\begin{aligned} \lim_{r\to\infty} r^{n+j+2\delta}\rho_0^{(j)}(r) &= (-1)^j\frac{\Gamma(n+2\delta)}{\Gamma(n+j+2\delta)}\lim_{r\to\infty} r^{n+2\delta}\rho_0(r) \\ &= (-1)^j\frac{\Gamma(n+2\delta)}{\Gamma(n+j+2\delta)}\left(\frac{\Gamma(\frac{n}{2}+\lceil s\rceil+3+\delta)}{\Gamma(\frac{n}{2}+\delta)}\right)^2|L_\infty|^2 \neq 0 \end{aligned}$$

for all $j \in [0, \lceil s\rceil + 3]$. In particular, $\rho_0^{(j)}(r)$ is exactly order $r^{-n-j-2\delta}$ as $r \to \infty$. Then, one sees from (2.7) that

$$\lim_{r\to\infty} r^{\frac{n}{2}-1}\sum_{j=0}^{k}\alpha_{j,k}r^j v_0^{(j)}(r) = 0$$

for all $k \in [1, \lceil s \rceil + 4]$, where $\alpha_{j,k}$ is the same constant as in (2.7). Since $\alpha_{k,k} = 1 \neq 0$, an induction argument proves that $\lim_{r\to\infty} r^{\frac{n}{2}+j-1} v_0^{(j)}(r)$ ($j = 0, 1, \ldots, \lceil s \rceil + 1$) exists. Once they exist, they satisfy

$$\begin{aligned}\lim_{r\to\infty} r^{\frac{n}{2}+j-1} v_0^{(j)}(r) &= (-1)^j \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}+j-1)} \lim_{r\to\infty} r^{\frac{n}{2}-1} v_0(r) \\ &= (-1)^j \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}+j-1)} \sqrt{\frac{2|\lambda|}{n-2} \int_0^\infty |A_0(r)|^2 r^{n-1} dr} \neq 0.\end{aligned}$$

As a result, there exist r_1 and constants c and C such that if $r \geqslant r_1$ then

$$\begin{aligned}cr^{-\frac{n}{2}-j-\delta} &\leqslant |A_0^{(j)}(r)| \leqslant Cr^{-\frac{n}{2}-j-\delta}, \\ cr^{-n-j-2\delta} &\leqslant |\rho_0^{(j)}(r)| \leqslant Cr^{-n-j-2\delta}, \\ cr^{-\frac{n}{2}+1-j} &\leqslant |v_0^{(j)}(r)| \leqslant Cr^{-\frac{n}{2}+1-j},\end{aligned} \tag{3.3}$$

for $j \in [0, \lceil s \rceil + 3]$ (the third inequality also holds for $j = \lceil s \rceil + 4$).

Step 2. We show $\mathbf{A}_0 \in H^{s+3}(\mathbb{R}^n)$. Note that $\mathbf{A}_0 \in L^2(\mathbb{R}^n)$ follows from (3.3) and (1.7). Moreover, for $k \in [1, \lceil s \rceil + 3]$, we have

$$\left\| \nabla^k \mathbf{A}_0 \right\|_{L^2(\mathbb{R}^n)} \leqslant C \sum_{j=1}^{k} \int_0^\infty |r^{j-\lceil s \rceil-3} A_0^{(j)}(r)|^2 r^{n-1} dr < \infty$$

thanks to (3.3) and (1.7).

Step 3. Let us prove that $\nabla \mathbf{\Phi}_0 \in Y^{s+4}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$. It follows from (3.1) and (3.3) that

$$\|\nabla \mathbf{\Phi}_0\|^p_{L^p(\mathbb{R}^n)} = \int_0^\infty |v_0(r)|^q r^{n-1} dr < \infty$$

if $(-n/2+1)p + n - 1 < -1$, that is, if $p > 2^*$. We next consider $\nabla(\nabla \mathbf{\Phi}_0)$. Note that

$$\|\nabla(\nabla \mathbf{\Phi}_0)\|^q_{L^q(\mathbb{R}^n)} \leqslant C \sum_{j=0}^{1} \int_0^\infty \left| r^{j-1} v_0^{(j)}(r) \right|^q r^{n-1} dr.$$

We deduce from (3.1), (3.2), and (3.3) that the right hand side is bounded if $-(n/2)q + n - 1 < -1$, that is, if $q > 2$. The proof of $\nabla^k(\nabla \mathbf{\Phi}_0) \in L^2(\mathbb{R}^n)$ ($k \in [2, \lceil s \rceil + 4]$) is similar. An elementary computation shows that

$$\left\| \nabla^k (\nabla \mathbf{\Phi}_0) \right\|_{L^2(\mathbb{R}^n)} \leqslant C \sum_{j=0}^{k} \int_0^\infty \left| \frac{v_0^{(j)}(r)}{r^{k-j}} \right|^2 r^{n-1} dr.$$

This is finite because of (3.2) and (3.3) and asumpton $\kappa \geqslant \lceil s \rceil + 3 - n/2$. $\square$

3.2. **Persistence of the regularity.** We next show that the global solution given in Theorem 2.2 keeps the same regularity as the initial data for all positive time, thanks to its explicit representation.

Proposition 3.2. *Let Assumption 1.1 be satisfied. Let* $(a_0(t,r), \phi_0(t,r))$ *be the global solution of* (2.4) *given by the formula* (2.6) *in Theorem 2.2. Then, the corresponding global solution*

$$\mathbf{a}(t,x) = a_0(t,|x|), \quad \mathbf{\Phi}(t,x) = \phi_0(t,|x|) \tag{3.4}$$

of (1.6) *satisfies* $(\mathbf{a}(t), \nabla\mathbf{\Phi}(t)) \in H^{s+3}(\mathbb{R}^n) \times Y^{s+4}_{(2^*,\infty],(2,\infty]}(\mathbb{R}^n)$ *for all* $t \geqslant 0$. *Moreover,* (1.14) *and* (1.15) *hold.*

Proof. First of all, we put $v_0(r) = \Phi_0'(r)$ and

$$F(R) := \frac{nv_0(R)}{2R} \geqslant 0, \quad G(R) := \frac{2|\lambda||A_0(R)|^2 R}{(n-2)v_0(R)} = v_0'(R) + \frac{(n-2)v_0(R)}{2R} \geqslant 0.$$

Then, they simplify the notation as follows:

$$\begin{aligned} a_0(t, X(t,R)) &= A_0(R)(1+F(R)t)^{-1/2}(1+G(R)t)^{-1/2}, \\ \partial_r\phi_0(t, X(t,R)) &= v_0(R)(1+F(R)t)^{2/n-1}, \\ X(t,R) &= R(1+F(R)t)^{2/n}, \\ \partial_R X(t,R) &= (1+F(R)t)^{2/n-1}(1+G(R)t). \end{aligned}$$

Step 1. Let us collect the properties of F and G. First is the decay rate as $R \to 0$. It follows from (3.1) and (3.2) that

$$F^{(j)}(R) = O(R^{\kappa-j}), \quad G^{(j)}(R) = O(R^{\kappa-j}) \tag{3.5}$$

as $R \to 0$ for all $j \in [0, \lceil s\rceil + 3]$ (the first estimate also holds for $j = \lceil s\rceil + 4$). Notice that (1.7) gives $G(R) > 0$ for all $R > 0$. Therefore, there exist R_0 and positive constants c and C such that

$$c \leqslant \frac{F(R)}{R^\kappa} \leqslant C, \quad c \leqslant \frac{G(R)}{R^\kappa} \leqslant C, \quad c \leqslant \frac{G(R)}{F(R)} \leqslant C \tag{3.6}$$

holds for $R \leqslant R_0$. A similar argument as in Step 1 of the proof of Proposition 3.1 shows that there exist R_1 and positive constants c and C such that

$$\begin{aligned} cR^{-\frac{n}{2}-j} &\leqslant |F^{(j)}(R)| \leqslant CR^{-\frac{n}{2}-j}, \\ cR^{-\frac{n}{2}-j-2\delta} &\leqslant |G^{(j)}(R)| \leqslant CR^{-\frac{n}{2}-j-2\delta} \end{aligned} \tag{3.7}$$

for $j \in [0, \lceil s\rceil + 3]$ if $R \geqslant R_1$ (the first inequality holds for $j = \lceil s\rceil + 4$).

Step 2-a. We show the uniform boundedness of $\nabla\mathbf{\Phi} \in Y^{s+1}_{p,q}(\mathbb{R}^n)$ in time for $p > 2^*$ $q > 2 + 4\delta/n$. Since $\sup_{r\geqslant 0}(|F(r)| + |G(r)|) < \infty$, we see that

$$\begin{aligned} \|\nabla\mathbf{\Phi}(t)\|^p_{L^p(\mathbb{R}^n)} &\leqslant \int_0^\infty |v_0(R)|^p R^{n-1}(1+F(R)t)^{1-\frac{2p}{2^*}}(1+G(R)t)dR \\ &\leqslant C\,\|\nabla\mathbf{\Phi}_0\|^p_{L^p(\mathbb{R}^n)} < \infty \end{aligned}$$

for any $t \geqslant 0$ and $p > 2^*$, where we have used the fact that $\frac{1+G(R)t}{1+F(R)t}$ is bounded uniformly in t and R. The L^∞ estimate is obtained in the similar way. By the Lebesgue convergence theorem, one sees that $\|\nabla\mathbf{\Phi}(t)\|_{L^p(\mathbb{R}^n)} \to 0$ as $t \to \infty$.

Step 2-b. We next estimate

$$\begin{aligned}
&\|\nabla(\nabla\mathbf{\Phi})(t)\|_{L^q(\mathbb{R}^n)}^q\\
&\leqslant C\int_0^\infty \left|\frac{v(t,X)}{X}\right|^q X^{n-1}\partial_R X dR + C\int_0^\infty |v'(t,X)|^q X^{n-1}\partial_R X dR\\
&\leqslant C\int_0^\infty |v_0(R)|^q R^{n-q-1}(1+F(R)t)^{1-q}(1+G(R)t)dR\\
&\quad + C\int_0^\infty |v_0'(R)|^q R^{n-1}(1+F(R)t)(1+G(R)t)^{1-q}dR\\
&\quad + C\int_0^\infty |v_0(R)|^q R^{n-1}(1+F(R)t)^{1-q}(1+G(R)t)^{1-q}|F'(R)|^q t^q dR
\end{aligned}$$

for $q > 4\delta/n$. We divide each time integrals as $\int_0^\infty = \int_0^{R_0} + \int_{R_0}^{R_1} + \int_{R_1}^\infty$. We easily handle the seconds and prove that they are order $o(1)$ as $t \to \infty$ since integrals are over a bounded interval. Hence, we left the detail and only establish uniform estimates of the integrals on the intervals $(0, R_0)$ and (R_1, ∞).

By (3.6), $\frac{1+F(R)t}{1+G(R)t}$ is bounded from below and above uniformly in time for $R \leqslant R_0$. Therefore,

$$\begin{aligned}
&\int_0^{R_0} |v_0(R)|^q R^{n-q-1}(1+F(R)t)^{1-q}(1+G(R)t)dR\\
&+\int_0^{R_0} |v_0'(R)|^q R^{n-1}(1+F(R)t)(1+G(R)t)^{1-q}dR\\
&\leqslant C\int_0^{R_0} R^{\kappa q+n-1}(1+F(R)t)^{2-q}dR.
\end{aligned}$$

Apparently, these two terms are order $o(1)$. Similarly, $\frac{1+F(R)t}{1+G(R)t} \leqslant CR^{2\delta}$ yields

$$\begin{aligned}
&\int_{R_1}^\infty |v_0(R)|^q R^{n-q-1}(1+F(R)t)^{1-q}(1+G(R)t)dR\\
&+\int_{R_1}^\infty |v_0'(R)|^q R^{n-1}(1+F(R)t)(1+G(R)t)^{1-q}dR\\
&\leqslant C\int_{R_1}^\infty R^{-\frac{n}{2}q+n-1+2\delta}dR < \infty
\end{aligned}$$

if $-\frac{n}{2}q + n + 2\delta < 0$, that is, if $q > 2 + 4\delta/n$. On the other hand,

$$\begin{aligned}
&\int_0^{R_0} |v_0(R)|^q R^{n-1}(1+F(R)t)^{1-q}(1+G(R)t)^{1-q}|F'(R)|^q t^q dR\\
&\leqslant C\int_0^{R_0} R^{\kappa q+n-1}\left(R^{\kappa q}(1+F(R)t)^{-q}t^q\right)dR.
\end{aligned}$$

It follows from the Young inequality that

$$R^{\kappa q}(1+F(R)t)^{2-2q}t^q \leqslant R^{\kappa q}t^q(CR^\kappa t)^{-q} \leqslant C.$$

Therefore, this term is also bounded. Similarly,

$$\int_{R_1}^{\infty} |v_0(R)|^q R^{n-1}(1+F(R)t)^{1-q}(1+G(R)t)^{1-q}|F'(R)|^q t^q dR$$
$$\leqslant C\int_{R_0}^{\infty} R^{-\frac{n}{2}q+n-1+2\delta}\left(R^{-\frac{n}{2}q}(1+F(R)t)^{2-2q}t^q\right)(1+G(R)t)^{2-q}dR.$$

This is uniformly bounded as $t\to\infty$ if $q>2+4\delta/n$ because the following estimate is true for $R\geqslant R_1$:

$$R^{-\frac{n}{2}q}(1+F(R)t)^{-q}t^q \leqslant C\left(\frac{R^{-\frac{n}{2}}t}{1+R^{-\frac{n}{2}}t}\right)^q.$$

Step 2-c. Finally, let us prove $\nabla^k(\nabla\mathbf{\Phi})\in L^2(\mathbb{R}^n)$ for $k\in[2,\lceil s\rceil+4]$. Note that

$$\left\|\nabla^k(\nabla\mathbf{\Phi})(t)\right\|^2_{L^2(\mathbb{R}^n)} \leqslant C\sum_{j=0}^{k}\int_0^\infty \left|\frac{\partial_r^j v(t,X(t,R))}{X(t,R)^{k-j}}\right|^2 X^{n-1}\partial_R X dR. \tag{3.8}$$

For simplicity, we define $v(t,r):=\partial_r\phi(t,r)$. By the explicit representations of v_0 and $\partial_R X$, we obtain

$$\begin{aligned}(\partial_r^k v)(t,X(t,R)) = &\sum_{l_i\geqslant 0, l_2<k, l_1+l_2+l_3\leqslant k}\ \sum_{m_i\in(\mathbb{N}\cup\{0\})^{l_i}, |m_1|+|m_2|=k-l_1-l_2-l_3} \\ &C_{k,l_i,m_1,m_2} v_0^{(l_3)}(R)(1+F(R)t)^{(k-1)\left(1-\frac{2}{n}\right)-l_1}(1+G(R)t)^{-k-l_2}t^{l_1+l_2} \\ &\prod_{i_1=1}^{l_1}F^{(1+m_{1i_1})}(R)\prod_{i_2=1}^{l_2}G^{(1+m_{2i_2})}(R).\end{aligned} \tag{3.9}$$

Therefore, our task is to prove that

$$\begin{aligned}\int_0^\infty &\frac{|v_0^{(l_3)}(R)|^2}{R^{2k-2j-n+1}}(1+F(R)t)^{2j+1-\frac{4k}{n}-2l_1}(1+G(R)t)^{1-2j-2l_2}t^{2l_1+2l_2} \\ &\times\prod_{i_1=1}^{l_1}F^{(1+m_{1i_1})}(R)^2\prod_{i_2=1}^{l_2}G^{(1+m_{2i_2})}(R)^2 dR<\infty\end{aligned} \tag{3.10}$$

for each $k\in[2,\lceil s\rceil+4]$, $j\in[0,k]$, $l_i\geqslant 0$ $(i=1,2,3)$ with $l_2<j$ and $l_1+l_2+l_3\leqslant j$, and $m_i\in(\mathbb{N}\cup\{0\})^{l_i}$ $(i=1,2)$ with $|m_1|+|m_2|=j-l_1-l_2-l_3$. We divide $\int_0^\infty=\int_0^{R_0}+\int_{R_0}^{R_1}+\int_{R_1}^\infty$ and denote the left hand side of (3.10) as $J_1+J_2+J_3$. As in Step 2-b, it is easy to see that J_2 is finite and tends to zero as $t\to\infty$.

We now estimate J_1. By (3.1), (3.5), and (3.6), J_1 is bounded by

$$C\int_0^{R_0}(R^{\kappa+1-l_3})^2R^{-2k+2j+n-1}\prod_{i_1=1}^{l_1}(R^{\kappa-1-m_{1i_1}})^2\prod_{i_2=1}^{l_2}(R^{\kappa-1-m_{2i_2}})^2dR$$
$$=C\int_0^{R_0}R^{2\kappa-2(k-1)+n+2\kappa(l_1+l_2)-1}dR<\infty$$

because $2\kappa-2(k-1)+n>2\kappa-2(\lceil s\rceil+3)+n>0$ by the choice of κ.

We finally treat J_3. From (3.7), it is bounded by

$$C\int_{R_1}^{\infty}\frac{R^{-n+2-2l_3-2\delta}}{R^{2k-2j-n+1}}\prod_{i_1=1}^{l_1}(R^{-\frac{n}{2}-1-m_{1i_1}})^2\prod_{i_2=1}^{l_2}(R^{-\frac{n}{2}-1-m_{2i_2}-2\delta})^2 \times\left(\frac{1+F(R)t}{1+G(R)t}\right)^{2j}(1+F(R)t)^{1-\frac{4k}{n}-2l_1}(1+G(R)t)^{1-2l_2}t^{2(l_1+l_2)}dR. \tag{3.11}$$

By Young's inequality and (3.7), we have

$$(1+F(R)t)^{1-\frac{4k}{n}-2l_1}(1+G(R)t)^{1-2l_2}t^{2(l_1+l_2)}\leqslant CR^{n(l_1+l_2)+2\delta\max(0,2l_2-\frac{4k}{n}+1)}.$$

Therefore, (3.11) is bounded by

$$C\int_{R_1}^{\infty}R^{-2(k-1)+\delta(4j-2-2\min(2l_2,\frac{4k}{n}-1))-1}dR,$$

which is finite if $-2(k-1)+\delta(4j-2-2\min(2l_2,\frac{4k}{n}-1))<0$. Since $j\in[0,k]$ and $l_2\in[0,j]$, this condition is satisfied for all j and l_2 if $-2(k-1)+\delta(4k-2-2\min(0,\frac{4k}{n}-1))<0$. The latter condition can be written as

$$\delta<\min\left(\frac{1}{2}-\frac{1}{4k-2},\frac{n}{2(n-2)}-\frac{n}{2k(n-2)}\right)\leqslant\min\left(\frac{1}{3},\frac{1}{4}+\frac{1}{2(n-2)}\right)$$

thanks to the fact that the range of k is $[2,\lceil s\rceil+4]$.

The estimate of $\mathbf{a}$ is similar and so we omit the detail. We only note that the L^2 norm is conserved: By the explicit representation of a_0,

$$\begin{aligned}\|\mathbf{a}(t)\|_{L^2(\mathbb{R}^n)}&=\int_0^{\infty}|a_0(t,X(t,R))|^2X(t,R)^{n-1}\partial_RX(t,R)dR\\&=\int_0^{\infty}|A_0(R)|^2R^{n-1}dR.\end{aligned}$$

□

Remark 3.3. The similar proof shows $\nabla\Phi\in Y^{s+4}_{p,q}$ also for $p>2^*$ and $q\in(2,2+4\delta/n]$ at the sacrifice of the uniform bound in time. We only have to replace the bounds of $1+F(R)t$ and $1+G(R)t$ with rougher ones:

$$1\leqslant 1+F(R)t,\ 1+G(R)t\leqslant 1+t\sup_R(|F(R)|+|G(R)|).$$

We note that also the assumption $A_0(r)\neq 0$ $(r>0)$ is needed only for the uniform boundedness in time.

4. Preliminaries for the proof of Theorem 1.10

4.1. **Some estimates.**

We first recall a consequence of the Hardy-Littlewood-Sobolev inequality, which can be found in [17, Th. 4.5.9] or [15, Lemma 7]:

Lemma 4.1. *If $\varphi\in\mathcal{D}'(\mathbb{R}^n)$ is such that $\partial_j\varphi\in L^p(\mathbb{R}^n)$ $(j=1,\dots,n)$ for some $p\in]1,n[$, then there exists a constant c such that $\varphi-c\in L^q(\mathbb{R}^n)$, with $1/p=1/q+1/n$.*

The next two lemmas can be found in [5, 18]:

Lemma 4.2 (Commutator estimate)**.** *Let $s \geqslant 0$ be a real number and $k \geqslant 0$ be an integer such that $k \leqslant s$. There exists $C > 0$ such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leqslant C(\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|\nabla^k f\|_{H^{s-k}} \|g\|_{L^\infty}).$$

Lemma 4.3. *Let $s \geqslant 0$ be a real number and $k \geqslant 0$ be an integer such that $k \leqslant s$. There exists $C > 0$ such that*

$$\|\Lambda^s(fg)\|_{L^2} \leqslant C(\|f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|\nabla^k g\|_{H^{s-k}}), \tag{4.1}$$

for all $f \in H^s \cap L^\infty$ and $g \in \dot{H}^k \cap \dot{H}^s \cap L^\infty$, and that

$$\|\Lambda^s \nabla(fg)\|_{L^2} \leqslant C(\|\nabla f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|\nabla g\|_{H^s}), \tag{4.2}$$

for all $f, g \in \dot{H}^1 \cap \dot{H}^s \cap L^\infty$.

The following lemma is a modification of the results in [2, 7]:

Lemma 4.4. *Let $n \geqslant 3$, $k \in \mathbb{R}_+$, and $s \in \mathbb{R}$. Let $\gamma \in (0, n)$ satisfy $\frac{n}{2} - k < \gamma \leqslant n - k$. Then, there exists C_s such that*

$$\left\| |\nabla|^k (|x|^{-\gamma} * f) \right\|_{H^s} \leqslant C_s(\|f\|_{H^s} + \|f\|_{L^1}), \quad \forall f \in L^1 \cap H^s.$$

Proof. Since $\mathcal{F}|x|^{-\gamma} = C|\xi|^{-n+\gamma}$ for $\gamma \in (0, n)$, it holds that

$$\left\| |\nabla|^k (|x|^{-\gamma} * f) \right\|_{H^s} = C \left\| \langle\xi\rangle^s |\xi|^{-n+\gamma+k} \mathcal{F} f \right\|_{L^2}.$$

The high frequency part ($|\xi| > 1$) is bounded by $C \|f\|_{H^s}$ if $-n + \gamma + k \leqslant 0$. On the other hand, the low frequency part ($|\xi| \leqslant 1$) is bounded by

$$C \|\mathcal{F} f\|_{L^\infty} \left(\int_{|\xi| \leqslant 1} |\xi|^{2(-n+\gamma+k)} d\xi \right)^{\frac{1}{2}} \leqslant C \|f\|_{L^1}$$

if $2(-n + \gamma + k) > -n$, that is, if $\gamma > n/2 - k$. $\square$

4.2. **Local existence of the unique solution of** (1.5)**.** We now give an existence result of the unique solution of (1.5). With slight generalization of the nonlinearity, let us consider the following system of Hartree type:

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon(0, x) = A_0^\varepsilon(x); \\ \partial_t \phi^\varepsilon + \dfrac{1}{2}|\nabla\phi^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2) = 0, & \phi^\varepsilon(0, x) = \Phi_0(x). \end{cases} \tag{4.3}$$

The system (1.5) corresponds to (4.3) with $\gamma = n - 2$.

Assumption 4.5. *Let $n \geqslant 3$ and $\max(n/2 - 2, 0) < \gamma \leqslant n - 2$. Let $\lambda \in \mathbb{R}$. We suppose the following conditions with some $s > n/2 + 1$:*
- *The initial amplitude $A_0^\varepsilon \in H^{s+1}(\mathbb{R}^n)$ uniformly in $\varepsilon \in (0, 1]$.*
- *The initial phase $\Phi_0 \in C^4(\mathbb{R}^n)$ satisfies $\nabla\Phi_0 \in Y^{s+2}_{p,q}(\mathbb{R}^n)$ for some $p \in (2^*, \infty]$ and $q \in (2, n)$ with $p \geqslant q$.*

Theorem 4.6. *Let Assumption 4.5 be satisfied. Then, there exists $T > 0$ independent of ε and s such that, for all $\varepsilon \in (0, 1]$, (4.3) has a unique solution*

$$(a^\varepsilon, \phi^\varepsilon) \in C([0, T]; H^{s+1}(\mathbb{R}^n) \times C^4(\mathbb{R}^n))$$

with $\nabla\phi^\varepsilon \in C([0, T]; Y^{s+2}_{p,q})$. Moreover, $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solves (1.2) and the solution $(a^\varepsilon, \nabla\phi^\varepsilon)$ is bounded in $L^\infty([0, T]; H^{s+1}(\mathbb{R}^n) \times Y^{s+2}_{p,q}(\mathbb{R}^n))$ uniformly in $\varepsilon \in (0, 1]$ and the following properties hold:

- $\phi^\varepsilon - \Phi_0 \in L^{\max(\frac{p}{2},\frac{n}{\gamma}+)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
- *If $n \geqslant 5$ and if p in Assumption 4.5 satisfies $2^* < p < n$, then $\Phi_0, \phi^\varepsilon \in L^\infty(\mathbb{R}^n)$.*

Remark 4.7. $2^* = 2n/(n-2) < n$ if and only if $n \geqslant 5$.

Denoting $v^\varepsilon := \nabla \phi^\varepsilon$, we obtain the following system:

$$\begin{cases} \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon_{|t=0} = A^\varepsilon_0, \\ \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \lambda \nabla \left(|x|^{-\gamma} * |a^\varepsilon|^2 \right) = 0, & v^\varepsilon_{|t=0} = \nabla \Phi_0. \end{cases} \tag{4.4}$$

We first solve this system. Then, as we seen below, we can reconstruct ϕ^ε from v^ε. The proof goes along the classical energy argument. Then, the main part of the proof is to establish an a priori estimate. We hence perform precisely only this part. As a first step, we show the following proposition.

Proposition 4.8. *Let Assumption 4.5 be satisfied. If $(a^\varepsilon, v^\varepsilon)$ is a solution of (4.4) in $C([0,T]; H^{s+1} \times Y^{s+2}_{p,q})$, then its "partial energy" $E_{\rm part}(t) := \|a^\varepsilon\|^2_{H^{s+1}} + \left\|\nabla^2 v^\varepsilon\right\|^2_{H^s}$ satisfies*

$$\frac{d}{dt} E_{\rm part}(t) \leqslant C E_{\rm part}(t)^{\frac{3}{2}}.$$

Proof. We first estimate the H^{s+1} norm of a^ε. We use the following convention for the scalar product in L^2:

$$\langle \varphi, \psi \rangle := \int_{\mathbb{R}^n} \varphi(x) \overline{\psi(x)} dx.$$

The notation $\Lambda = (1-\Delta)^{1/2}$ is also used. Then,

$$\frac{d}{dt} \|a^\varepsilon\|^2_{H^{s+1}} = 2 \operatorname{Re} \left\langle \partial_t \Lambda^{s+1} a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle.$$

Let us bound the right hand side with the relation

$$\partial_t \Lambda^{s+1} a^\varepsilon + \Lambda^{s+1}(v^\varepsilon \cdot \nabla a^\varepsilon) + \frac{1}{2}\Lambda^{s+1}(a^\varepsilon \nabla \cdot v^\varepsilon) - i\frac{\varepsilon}{2}\Delta \Lambda^{s+1} a^\varepsilon = 0.$$

This part is standard (for details, see [2, 7]). The point is that we cannot not use $\|\nabla v^\varepsilon\|_{L^2}$ as a bound. This is done by the use of Lemma 4.2 and (4.1) with suitable k. For example, Lemma 4.2 with $k=2$ shows the estimate

$$\begin{aligned} &| \operatorname{Re} \left\langle [\Lambda^{s+1}, v^\varepsilon] \cdot \nabla a^\varepsilon, \Lambda^{s+1} a^\varepsilon \right\rangle | \\ &\qquad \leqslant C(\|\nabla v^\varepsilon\|_{L^\infty} \|\nabla a^\varepsilon\|_{H^s} + \left\|\nabla^2 v^\varepsilon\right\|_{H^{s-1}} \|\nabla a^\varepsilon\|_{L^\infty}) \|a^\varepsilon\|_{H^{s+1}}, \end{aligned}$$

in which $\|\nabla v^\varepsilon\|_{L^2}$ does not appear. As a result, we obtain

$$\frac{d}{dt} \|a^\varepsilon\|^2_{H^{s+1}} \leqslant C(\|a^\varepsilon\|_{W^{1,\infty}} + \|\nabla v^\varepsilon\|_{L^\infty})(\|a^\varepsilon\|_{H^{s+1}} + \left\|\nabla^2 v^\varepsilon\right\|_{H^s}) \|a^\varepsilon\|_{H^{s+1}}.$$

Recall that $v^\varepsilon \in Y^{s+2}_{p,q}$ and so that $\nabla v^\varepsilon \to 0$ as $|x| \to \infty$ by the definition of Y. Hence, by the Sobolev embedding, $\|\nabla v^\varepsilon\|_{L^\infty} \leqslant C \left\|\nabla^2 v^\varepsilon\right\|_{H^s}$. We end up with

$$\frac{d}{dt} \|a^\varepsilon\|^2_{H^{s+1}} \leqslant C E_{\rm part}(t)^{\frac{3}{2}}. \tag{4.5}$$

Let us proceed to the estimate of v^ε. We denote the operator $\Lambda^s\nabla^2$ by Q. We deduce from the equation for v^ε that

$$\partial_t Qv^\varepsilon + Q(v^\varepsilon\cdot\nabla v^\varepsilon) + Q\nabla(|x|^{-\gamma}*|a^\varepsilon|^2) = 0. \tag{4.6}$$

We consider the coupling of this equation and Qv^ε. The coupling with the second term of the left hand side of (4.6) can be written as

$$\begin{aligned}\langle Q(v^\varepsilon\cdot\nabla v^\varepsilon), Qv^\varepsilon\rangle &= \langle v^\varepsilon\cdot\nabla Qv^\varepsilon, Qv^\varepsilon\rangle + \left\langle[\Lambda^s\nabla, v^\varepsilon]\cdot\nabla^2 v^\varepsilon, Qv^\varepsilon\right\rangle\\ &\quad+\langle\Lambda^s\nabla(\nabla v^\varepsilon\cdot\nabla v^\varepsilon), Qv^\varepsilon\rangle.\end{aligned}$$

As the previous case, integration by parts shows

$$|\operatorname{Re}\langle v^\varepsilon\cdot\nabla Qv^\varepsilon, Qv^\varepsilon\rangle| \leqslant \frac{1}{2}\|\nabla v^\varepsilon\|_{L^\infty}\left\|\nabla^2 v^\varepsilon\right\|_{H^s}, \tag{4.7}$$

and the commutator estimate with $k=1$ also shows

$$\begin{aligned}&|\operatorname{Re}\left\langle[\Lambda^s\nabla, v^\varepsilon]\cdot\nabla^2 v^\varepsilon, Qv^\varepsilon\right\rangle|\\ &\quad\leqslant C(\|\nabla v^\varepsilon\|_{L^\infty}\left\|\nabla^2 v^\varepsilon\right\|_{H^s} + \left\|\nabla^2 v^\varepsilon\right\|_{H^{s-1}}\left\|\nabla^2 v^\varepsilon\right\|_{L^\infty})\left\|\nabla^2 v^\varepsilon\right\|_{H^s}.\end{aligned} \tag{4.8}$$

We also have

$$|\operatorname{Re}\langle\Lambda^s\nabla(\nabla v^\varepsilon\cdot\nabla v^\varepsilon), Qv^\varepsilon\rangle| \leqslant C\|\nabla v^\varepsilon\|_{L^\infty}\left\|\nabla^2 v^\varepsilon\right\|_{H^s}\left\|\nabla^2 v^\varepsilon\right\|_{H^s} \tag{4.9}$$

by (4.2). For the estimate of the Hartree nonlinearity, we use Lemma 4.4 with $k=2$ to obtain

$$\begin{aligned}\left\|\lambda\nabla^3(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{H^s} &\leqslant C\left\|\nabla^2(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{H^{s+1}}\\ &\leqslant C(\|a^\varepsilon\|_{L^\infty}\|a^\varepsilon\|_{H^{s+1}} + \|a^\varepsilon\|_{L^2}^2)\end{aligned} \tag{4.10}$$

as long as $\gamma\in(n/2-2.n-2]$. Sum up (4.6)–(4.10) to conclude that

$$\frac{d}{dt}\left\|\nabla^2 v^\varepsilon\right\|_{H^s}^2 \leqslant CE_{\mathrm{part}}(t)^{\frac{3}{2}}, \tag{4.11}$$

which completes the proof. □

We now prove Theorem 4.6.

Proof of Theorem 4.6. We first obtain the solution $(a^\varepsilon, v^\varepsilon)$ of (4.4) by the energy method and then integrate v^ε to construct ϕ^ε.

Step 1. We shall show the existence of the solution $(a^\varepsilon, v^\varepsilon)\in C([0,T]; H^{s+1}\times Y_{p,q}^{s+2})$ of (4.4) for small $T>0$. Let us derive an a priori estimate of the energy

$$E(t) := \|a^\varepsilon(t)\|_{H^{s+1}}^2 + \|v^\varepsilon(t)\|_{Y_{p,q}^s}^2. \tag{4.12}$$

By Proposition 4.8 and Gronwall's lemma, there exists T such that

$$\sup_{t\in[0,T]} E_{\mathrm{part}}(t) \leqslant 2E_{\mathrm{part}}(0). \tag{4.13}$$

Next we estimate v^ε and ∇v^ε. By the second equation of (4.4), we obtain

$$v^\varepsilon(t) = \nabla\Phi_0 - \int_0^t \left((v^\varepsilon\cdot\nabla)v^\varepsilon + \lambda\nabla(|x|^{-\gamma}*|a^\varepsilon|^2)\right)ds.$$

Therefore, we deduce by the Hölder inequality that

$$\begin{aligned}\|v^\varepsilon\|_{L^\infty([0,T];L^p)} &\leqslant \|\nabla\Phi_0\|_{L^p} + T\|v^\varepsilon\|_{L^\infty([0,T];L^p)}\|\nabla v^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)}\\ &\quad+ T|\lambda|\left\|\nabla(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^\infty([0,T];L^p)}\end{aligned} \tag{4.14}$$

and

(4.15)
$$\begin{aligned}\|\nabla v^\varepsilon\|_{L^\infty([0,T];L^q)} \leqslant{}& \left\|\nabla^2\Phi_0\right\|_{L^q} + T\left\|\nabla v^\varepsilon\right\|_{L^\infty([0,T];L^q)}\left\|\nabla v^\varepsilon\right\|_{L^\infty([0,T]\times\mathbb{R}^n)}\\ &+ T\left\|v^\varepsilon\right\|_{L^\infty([0,T];L^p)}\left\|\nabla^2 v^\varepsilon\right\|_{L^\infty([0,T];L^{\frac{pq}{p-q}})}\\ &+ T|\lambda|\left\|\nabla^2(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^\infty([0,T];L^q)}.\end{aligned}$$

Notice that $H^s \hookrightarrow L^2\cap L^\infty \hookrightarrow L^{\frac{pq}{p-q}}$ since $pq/(p-q)\in(2,\infty]$ holds by assumption $p\geqslant q>2$. Moreover, we infer from Lemma 4.4 that

$$\left\|\nabla(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^p} \leqslant C\left\||\nabla|^{1+n(\frac12-\frac1p)}(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^2} \leqslant C\left\|a^\varepsilon\right\|_{H^{s+1}}^2,$$
$$\left\|\nabla^2(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^q} \leqslant C\left\||\nabla|^{2+n(\frac12-\frac1q)}(|x|^{-\gamma}*|a^\varepsilon|^2)\right\|_{L^2} \leqslant C\left\|a^\varepsilon\right\|_{H^{s+1}}^2,$$

provided $n/p-1<\gamma\leqslant n-2$ and $n/q-2<\gamma\leqslant n-2$, respectively. By the assumptions $p>2^*$ and $q>2$, $\max(n/p-1,n/q-2)<n/2-2$. Letting T so small that $T(2E_{\mathrm{part}}(0))<1/3$ if necessary, one sees from (4.13) that

(4.16)
$$\|v^\varepsilon\|_{L^\infty([0,T];L^p)}+\|\nabla v^\varepsilon\|_{L^\infty([0,T];L^q)} \leqslant 3\|\nabla\Phi_0\|_{L^p}+3\left\|\nabla^2\Phi_0\right\|_{L^q}+C(E_{\mathrm{part}}(0)).$$

Plugging (4.16) to (4.13), we obtain the desired energy estimate: There exist T and C such that $\sup_{t\in[0,T]}E(t)\leqslant C(E(0))$. Thus, we see from a standard argument that a solution $(a^\varepsilon,v^\varepsilon)$ of (4.4) exists in $C([0,T];H^{s+1}\times Y^{s+2}_{p,q})$.

Step 2. We next investigate the decay property of v^ε and show the uniqueness of the solution of (4.4). Since $q<n$ by assumption, $v^\varepsilon\in Y^{s+2}_{p,q}(\mathbb{R}^n)\hookrightarrow L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. By the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

(4.17)
$$v^\varepsilon-\nabla\Phi_0 = -\int_0^t\left((v^\varepsilon\cdot\nabla)v^\varepsilon+\lambda\nabla(|x|^{-\gamma}*|a^\varepsilon|^2)\right)ds \in L^{\max\left(\frac{pq}{p+q},\frac{n}{\gamma+1}+\right)}\cap L^\infty.$$

Notice that $pq/(p+q)<p$ for all $p>2^*$ and that $n/(\gamma+1)<\min(2^*,n)<p$ since $\gamma>\max(n/2-2,0)$. Therefore, $v^\varepsilon-\nabla\Phi_0$ decreases at spatial infinity faster than v^ε and $\nabla\Phi_0$ themselves.

Let us proceed to the uniqueness of (4.4). Let $(a_1^\varepsilon,v_1^\varepsilon)$ and $(a_2^\varepsilon,v_2^\varepsilon)$ be two solutions of (4.4) in $C([0,T];H^{s+1}\times Y^{s+2}_{p,q})$ with $(a_i^\varepsilon,v_i^\varepsilon)(0)=(A_0^\varepsilon,\nabla\Phi_0)$. Put $d_a^\varepsilon=a_1^\varepsilon-a_2^\varepsilon$ and $d_v^\varepsilon=v_1^\varepsilon-v_2^\varepsilon$. We remark that $d_a^\varepsilon(0)\equiv0$ and $d_v^\varepsilon(0)\equiv0$. Moreover, we see from the above estimate (4.17) that $d_v^\varepsilon=(v_1^\varepsilon-\nabla\Phi_0)-(v_2^\varepsilon-\nabla\Phi_0)$ and so $d_v^\varepsilon\to0$ as $|x|\to\infty$. Now, we estimate

$$E_d(t):=\|d_a^\varepsilon(t)\|_{L^2}^2+\|\nabla d_v^\varepsilon(t)\|_{L^2}^2.$$

It is important to note that ∇v_1^ε and ∇v_2^ε do not necessarily belong to L^2 by definition of $Y^s_{p,q}$. Nevertheless, their difference d_v^ε may do so because it is identically zero and so belongs to L^2 at the initial time. By an energy estimate, we have

$$\frac{d}{dt}E_d(t)\leqslant C(\|a_i^\varepsilon\|_{H^{s+1}},\|v_i^\varepsilon\|_{Y^{s+2}_{p,q}})E_d(t). \tag{4.18}$$

Hence, we conclude from Gronwall's lemma that

$$E_d(t)\leqslant C(\|a_i^\varepsilon\|_{H^{s+1}},\|v_i^\varepsilon\|_{Y^{s+2}_{p,q}})E_d(0)=0$$

as long as the solutions $(a_i^\varepsilon, v_i^\varepsilon)$ exist. This implies that $d_a^\varepsilon \equiv 0$ and $\nabla d_v^\varepsilon \equiv 0$. In particular, there exists a function $d = d(t)$ of time such that $d_v^\varepsilon(t,x) = d(t)$. Recall that $d_v^\varepsilon(t,x) \to 0$ as $|x| \to \infty$. As a result, $d(t) \equiv 0$ follows and we hence obtain $(a_1^\varepsilon, v_1^\varepsilon) = (a_2^\varepsilon, v_2^\varepsilon)$.

Step 3. We finally construct ϕ^ε such that $\nabla\phi^\varepsilon = v^\varepsilon$. Define ϕ^ε by

$$\phi^\varepsilon(t) = \Phi_0 - \int_0^t \left(\frac{1}{2}|v^\varepsilon(s)|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)(s)\right) ds.$$

Then, one easily verifies that $(a^\varepsilon, \nabla\phi^\varepsilon)$ solves (4.4) and that $\nabla\phi^\varepsilon \in Y_{p,q}^{s+2}$. Since we have already known the uniqueness of the solution to (4.4), $\nabla\phi^\varepsilon = v^\varepsilon$. Thus, $(a^\varepsilon, \phi^\varepsilon)$ is a unique solution to (4.3). Though ϕ^ε and Φ_0 themselves do not necessarily belong to any Lebesgue space, it follows from the Hölder inequality and the Hardy-Littlewood-Sobolev inequality that

$$\phi^\varepsilon(t) - \Phi_0 = -\int_0^t \left(\frac{1}{2}|v^\varepsilon|^2 + \lambda(|x|^{-\gamma} * |a^\varepsilon|^2)\right) ds \in L^{\max\left(\frac{p}{2}, \frac{n}{\gamma}+\right)} \cap L^\infty.$$

Moreover, it is bounded uniformly in $\varepsilon \in [0,1]$.

If $n \geqslant 5$ and $2^* < p < n$ then, applying Lemma 4.1, we see that there exists a constant $c_0 \in \mathbb{R}$ such that $\|\Phi_0 - c_0\|_{L^{p^*}} \leqslant C\|\nabla\Phi_0\|_{L^p}$. Moreover, since $\Phi_0 - c_0$ decays at the spatial inifinity, it follows by the Sobolev inequality that $\|\Phi_0 - c_0\|_{L^\infty} \leqslant C\left\|\nabla^2\Phi_0\right\|_{H^s}$, which shows $\Phi_0 \in L^\infty$ and so $\phi^\varepsilon \in L^\infty$. Remark that $p/2 < (p <)p^*$ and that $n/\gamma < n/(n/2-2) = 2^{**} < p^*$ since $n \geqslant 5$ and $p > 2^*$. Therefore, the difference $\phi^\varepsilon - \Phi_0$ decays faster than ϕ^ε and Φ_0. □

Remark 4.9. In [2, 7], the key for existence result is to solve the system for $(a^\varepsilon, \nabla v^\varepsilon)$ in $H^s \times H^s$ first. Here, we first solve the system for $(a^\varepsilon, \nabla^2 v^\varepsilon)$ in $H^{s+1} \times H^s$. This is the difference. The point is that even if $\nabla v^\varepsilon \notin L^2$, we obtain the energy estimate by the Sobolev embedding: If $n \geqslant 3$ and $\nabla v^\varepsilon \to 0$ as $|x| \to \infty$ then $\|\nabla v^\varepsilon\|_{L^\infty} \leqslant C\left\|\nabla^2 v^\varepsilon\right\|_{H^s}$. We also note that it would be difficult to solve the system for $(a^\varepsilon, \nabla^3 v^\varepsilon)$ in $H^{s+2} \times H^s$ if $n = 3, 4$ because $\nabla^3 v^\varepsilon \in H^s$ do not yield this kind of bound on ∇v^ε, in general.

5. Proof of Theorem 1.10

We prove the following theorem.

Theorem 5.1. *Let Assumption 1.8 be satisfied. Let $(a^\varepsilon, \phi^\varepsilon)$ be the solution to* (1.5) *given in Theorem 4.6. If* (1.6) *has a global solution (a_0, ϕ_0) which satisfies $\eta(T; s, p, q) < \infty$ for all $T < \infty$, then there exist*

$$(a_j, \phi_j) \in C([0,\infty); H^{s-2j+3} \times Y_{2^*,2}^{s-2j+5})$$

$(1 \leqslant j \leqslant N)$ and constant C_s depending only on n and s such that, for any $T > 0$, $(a^\varepsilon, \phi^\varepsilon)$ exists until $t = T$ and it holds that

$$\begin{cases} a^\varepsilon = a_0 + \displaystyle\sum_{j=1}^N \varepsilon^j a_j + O(\varepsilon^{N+1}) & \textit{in } L^\infty([0,T], H^{s-2N+1}(\mathbb{R}^n)), \\ \phi^\varepsilon = \phi_0 + \displaystyle\sum_{j=1}^N \varepsilon^j \phi_j + O(\varepsilon^{N+1}) & \textit{in } L^\infty([0,T], Y_{2^*,2}^{s-2N+3}(\mathbb{R}^n)) \end{cases}$$

for $\varepsilon \leqslant C\eta(T)e^{-3C_s\eta(T)T}$.

Remark 5.2. We note that ϕ^ε itself does not necessarily belong to the space $Y^{s+3}_{2^*,2}(\mathbb{R}^n)$ as shown in Theorem 4.6.

Remark 5.3. We will see from the following proof that ϕ_j $(j \geqslant 1)$ belongs to $C([0,\infty); Y^{s-2j+5}_{[\frac{2p}{2+p},\infty]\cap(1^{**},\infty],2})$ and the above expansion of ϕ^ε is valid in $C([0,\infty); Y^{s-2N+3}_{[\frac{2p}{2+p},\infty]\cap(1^{**},\infty],2})$. Remark that if $n \geqslant 5$ then $Y^{s}_{[\frac{2p}{2+p},\infty]\cap(1^{**},\infty],2} \subset H^s$.

Theorem 1.10 immediately follows from this theorem. Notice that the main amplitude β_0 is not a_0 but $a_0e^{i\phi_1}$. There is an interaction between the amplitude part and the phase part because of the presence of nonlinearity. This fact also leads us to some ill-posedness results for the "usual", that is, non-scaled nonlinear Schrödinger equations ([3, 7, 25]). Similarly, the function β_j is defined by (a_i, ϕ_i) $(i = 0, 1, \cdots, j+1)$. We will see that (a_j, ϕ_j) $(j \geqslant 1)$ solves the "j-th linearized system" of (1.5):

$$\begin{cases} \partial_t a_j + \displaystyle\sum_{i_1+i_2=j} \nabla\phi_{i_1}\cdot\nabla a_{i_2} + \sum_{i_1+i_2=j} \frac{1}{2} a_{i_1}\Delta\phi_{i_2} - i\frac{1}{2}\Delta a_{j-1} = 0, \\ \partial_t \phi_j + \displaystyle\sum_{i_1+i_2=j} \frac{1}{2}\nabla\phi_i\cdot\nabla\phi_j + \lambda \sum_{i_1+i_2=j} (|x|^{-(n-2)} * \mathrm{Re}(a_{i_1}\overline{a_{i_2}})) = 0. \\ a_j(0) = A_j, \quad \phi_j(0) = 0. \end{cases} \tag{5.1}$$

We separate the proof of Theorem 5.1 into three steps,

zeroth order: estimate on $a^\varepsilon - a_0$ and $\phi^\varepsilon - \phi_0$,
first order: estimate on $a^\varepsilon - a_0 - \varepsilon a_1$ and $\phi^\varepsilon - \phi_0 - \varepsilon\phi_1$,
higher order: estimate on $a^\varepsilon - \sum_{j=0}^k \varepsilon^k a_k$ and $\phi^\varepsilon - \sum_{j=0}^k \varepsilon^k \phi_k$ for $k \geqslant 2$.

However, we only prove the third step because, in order to exclude the dependency of C_s on the expansion level N, the main step is the third part. This constant C_s is chosen later (in the proof of Proposition 5.8, below).

5.1. **Proof the theorem – part 1: the zeroth order.**

We first state the estimate on the differences $a^\varepsilon - a_0$ and $\phi^\varepsilon - \phi_0$.

Proposition 5.4. *Let Assumption 1.8 be satisfied. Let* $(a^\varepsilon, \phi^\varepsilon)$ *be the solution to* (1.5) *given by Theorem 4.6 and* (a_0, ϕ_0) *be the global solution to* (1.6) *with* $\eta(T) < \infty$ *for all* $T < \infty$. *Then, there exists a constant* C_s *depending on* n *and* s, *and* Γ_1 *depending on* A_0^ε *such that, for any large* T,

$$\|a^\varepsilon - a_0\|_{L^\infty([0,T],H^{s+1})} + \|\nabla\phi^\varepsilon - \nabla\phi_0\|_{L^\infty([0,T],H^{s+2})} \leqslant \varepsilon\Gamma_1 e^{C_s\eta(T)T} \tag{5.2}$$

holds for all $\varepsilon \leqslant \varepsilon_0(T) \leqslant \eta(T)Ce^{-C_s\eta(T)T}$. *In particular, the existence time* T *of* $(a^\varepsilon, \phi^\varepsilon)$ *can be chosen so that* $\varepsilon \sim \eta(T)e^{-C_s\eta(T)T}$.

The proof proceeds as in [20].

5.2. **Proof of the theorem – part 2: the first order.** We next claim the following two points: First is that (a_1, ϕ_1) is defined globally in time as a limit $\varepsilon \to 0$ of $(\widetilde{a}_0^\varepsilon, \widetilde{\phi}_0^\varepsilon)$ (Proposition 5.5). Second is that the asymptotics

$$a^\varepsilon = a_0 + \varepsilon a_1 + O(\varepsilon^2), \quad v^\varepsilon = v_0 + \varepsilon v_1 + O(\varepsilon^2)$$

holds for large time (Proposition 5.6).

Proposition 5.5. *Let Assumption 1.8 be satisfied. Suppose that* (1.6) *has a global solution* (a_0, ϕ_0) *which satisfies* $\eta(T) < \infty$ *for all* $T < \infty$*. Then, there exists* $(a_1, \phi_1) \in C([0,\infty), H^{s+1} \times Y^{s+3}_{2^*,2})$ *which solves* (5.1)*. Let* $E_1(t) := \|a_1(t)\|_{H^{s+1}} + \|\nabla \phi_1(t)\|_{H^{s+2}}$*. Then, for any* $T > 0$*, we have the following bound*

$$\sup_{t\in[0,T]} E_1(t) \leqslant \Gamma_1 e^{C_s \eta(T) T} =: \eta_1(T), \tag{5.3}$$

where Γ_1*,* C_s*, and* η *are the same as in Proposition 5.4.*

Proposition 5.6. *Let Assumption 1.8 be satisfied. Let* $(a^\varepsilon, \phi^\varepsilon)$ *be the solution to* (1.5) *given by Theorem 4.6. Suppose that* (1.6) *has a global solution* (a_0, ϕ_0) *which satisfies* $\eta(T) < \infty$ *for all* $T < \infty$*. Let* (a_1, ϕ_1) *be the limit defined in Proposition 5.5. Let* C_s *be the same one as in Proposition 5.4. Then, there exists a constant* Γ_2 *depending on* A_0^ε *such that*

$$\begin{aligned} &\|a^\varepsilon - a_0 - \varepsilon a_1\|_{L^\infty([0,T],H^{s-1})} + \|\nabla(\phi^\varepsilon - \phi_0 - \varepsilon\phi_1)\|_{L^\infty([0,T],H^s)} \\ &\qquad \leqslant \varepsilon^2 \Gamma_2 \eta(T)^{-1} e^{3C_s\eta(T)T} e^{\varepsilon C_s \eta_1(T) T} \end{aligned} \tag{5.4}$$

holds for all $0 < \varepsilon \leqslant \varepsilon_1(T) \leqslant Ce^{-2C_s\eta(T)T}$*. In particular, the existence time* T *of* $(a^\varepsilon, \phi^\varepsilon)$ *can be chosen so that* $\varepsilon \sim e^{-2C_s\eta(T)T}$*.*

They are shown as in Propositions 5.7 and 5.8, below, respectively.

5.3. **Proof of the theorem – part 3: higher order.** We now consider the higher order expansion. Assume that the constant N in Assumption 1.8 is bigger than one. It is because if $N = 1$ then the proof of Theorem 5.1 is already finished with Proposition 5.6. The proof is based on the induction argument. We make following notation and definitions: Our goal is to show that the asymptotics

$$\begin{cases} a^\varepsilon = a_0 + \displaystyle\sum_{j=1}^m \varepsilon^j a_j + O(\varepsilon^{m+1}) \quad \text{in } L^\infty([0,T], H^{s-2m+1}(\mathbb{R}^n)), \\ \phi^\varepsilon = \phi_0 + \displaystyle\sum_{j=1}^m \varepsilon^j \phi_j + O(\varepsilon^{m+1}) \quad \text{in } L^\infty([0,T], Y^{s-2m+3}_{2^*,2}(\mathbb{R}^n)) \end{cases} \tag{5.5}$$

for $m = N$. We define the following function:

$$\eta_j(T) := \frac{\Gamma_j}{\eta(T)^{j-1}} e^{(2j-1)C_s\eta(T)T} \tag{5.6}$$

with $\eta(T)$ is an increasing function defined in (1.13), Γ_1 and Γ_2 are as in Propositions 5.4 and 5.6, respectively, and Γ_j $(j \geqslant 3)$ is a constant depending only on A_0^ε to be chosen later. Note that if $T \gg 1$ then

$$\eta_m(T) \gg \eta_{m-1}(T) \gg \cdots \gg \eta_1(T) \gg \eta(T) > 0.$$

The following two propositions complete the proof of Theorem 5.1.

Proposition 5.7. *Let Assumption 1.8 be satisfied for some $N \geqslant 2$. Suppose that (1.6) has a global solution (a_0, ϕ_0) which satisfies $\eta(T; s, p, q) < \infty$ for all $T < \infty$. Fix $k_0 \in [1, N-1]$. Assume that*

$$(a_j, \phi_j) \in C([0, \infty); H^{s-2j+3} \times Y^{s-2j+5}_{2^*,2})$$

$(1 \leqslant j \leqslant k_0)$ exist and all of them solve (5.1). We further assume that there exists a positive constant Γ_{k_0+1} such that

$$\varlimsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0} \varepsilon^j a_j}{\varepsilon^{k_0+1}} \right\|_{H^{s-2k_0+1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0} \varepsilon^j \phi_j)}{\varepsilon^{k_0+1}} \right\|_{H^{s-2k_0+2}} \right)$$

is bounded by $\eta_{k_0+1}(T)$ defined in (5.6) for any fixed $T > 0$. Then, there exists $(a_{k_0+1}, \phi_{k_0+1}) \in C([0, \infty); H^{s-2k_0+1} \times Y^{s-2k_0+3}_{2^*,2})$ which solves (5.1) and satisfies

$$\sup_{t\in[0,T]} \left(\|a_{k_0+1}\|_{H^{s-2k_0+1}} + \|\nabla\phi_{k_0+1}\|_{H^{s-2k_0+2}} \right) \leqslant \eta_{k_0+1}(T).$$

Proposition 5.8. *Let Assumption 1.8 be satisfied for some $N \geqslant 2$. Suppose that (1.6) has a global solution (a_0, ϕ_0) which satisfies $\eta(T; s, p, q) < \infty$ for all $T < \infty$. Fix $k_0 \in [1, N-1]$. Assume that, for all $1 \leqslant j \leqslant k_0 + 1$, the solution $(a_j, \phi_j) \in C([0, \infty); H^{s-2j+3} \times Y^{s-2j+5}_{2^*,2})$ of (5.1) exists and satisfies*

$$\sup_{t\in[0,T]} \left(\|a_j\|_{H^{s-2j+3}} + \|\nabla\phi_j\|_{H^{s-2j+4}} \right) \leqslant \eta_j(T).$$

Then, for any fixed $T > 0$,

$$\sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j a_j}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0-1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j \phi_j)}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0}} \right)$$

is bounded uniformly in $\varepsilon \in (0, \varepsilon_{k_0+2}]$. In particular, the asymptotics (5.5) holds with $m = k_0 + 1$ for $\varepsilon \in (0, \varepsilon_{k_0+2}]$. ε_{k_0+2} can be chosen so that $\varepsilon_{k_0+2} \leqslant C\eta(T)e^{-3C_s\eta(T)T}$. Moreover, there exists a positive constant Γ_{k_0+2} depending only on A_0^ε such that $\eta_{k_0+2}(T)$ defined in (5.6) bounds

$$\varlimsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \left(\left\| \frac{a^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j a_j}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0-1}} + \left\| \frac{\nabla(\phi^\varepsilon - \sum_{j=0}^{k_0+1} \varepsilon^j \phi_j)}{\varepsilon^{k_0+2}} \right\|_{H^{s-2k_0}} \right)$$

for any fixed large $T > 0$.

Proposition 5.6 implies that the assumption of Proposition 5.7 is satisfied for $k_0 = 1$. Then, we see by induction that Proposition 5.8 holds for $k_0 = N - 1$. Then, this gives (5.5) with $m = N$. Before the proof, we introduce some more notation. We write

$$b_m^\varepsilon = \frac{a^\varepsilon - \sum_{j=0}^m \varepsilon^j a_j}{\varepsilon^{m+1}}, \qquad w_m^\varepsilon = \frac{\nabla\phi^\varepsilon - \sum_{j=0}^m \varepsilon^j \nabla\phi_j}{\varepsilon^{m+1}}.$$

An elementary computation shows that $(b^\varepsilon_m, w^\varepsilon_m)$ satisfies

$$\begin{aligned}(5.7)\quad &\partial_t b^\varepsilon_m + \varepsilon^{m+1}\left(w^\varepsilon_m\cdot\nabla b^\varepsilon_m + \frac{1}{2}b^\varepsilon_m\nabla\cdot w^\varepsilon_m\right)\\ &+\sum_{\ell=0}^{m}\varepsilon^\ell\left(w^\varepsilon_m\cdot\nabla a_\ell + v_\ell\cdot\nabla b^\varepsilon_m + \frac{1}{2}b^\varepsilon_m\nabla\cdot v_\ell + \frac{1}{2}a_\ell\nabla\cdot v^\varepsilon_m\right)\\ &+\sum_{\ell=0}^{m-1}\varepsilon^\ell\sum_{i,j\leqslant m, i+j=m+1+\ell}\left(v_i\cdot\nabla a_j + \frac{1}{2}a_i\nabla\cdot v_j\right) - i\frac{1}{2}\Delta a_m = i\frac{\varepsilon}{2}\Delta b^\varepsilon_m,\end{aligned}$$

$$\begin{aligned}(5.8)\quad &\partial_t w^\varepsilon_m + \varepsilon^{m+1}\left(w^\varepsilon_m\cdot\nabla w^\varepsilon_m + \lambda\nabla(|x|^{-(n-2)} * |b^\varepsilon_m|^2)\right)\\ &+\sum_{\ell=0}^{m}\varepsilon^\ell((w^\varepsilon_m\cdot\nabla v_\ell + v_\ell\cdot\nabla w^\varepsilon_m) + \lambda\nabla(|x|^{-(n-2)} * \mathrm{Re}(a_\ell\overline{b^\varepsilon_m}))\\ &+\sum_{\ell=0}^{m-1}\varepsilon^\ell\sum_{i,j\leqslant m, i+j=m+1+\ell}\left(v_i\cdot\nabla v_j + \lambda\nabla(|x|^{-(n-2)} * \mathrm{Re}(a_i\overline{a_j}))\right) = 0,\end{aligned}$$

and

$$(5.9)\qquad b^\varepsilon_m(0) = \sum_{j=0}^{k-1-m}\varepsilon^j A_{j+m+1} + \varepsilon^{k-m} r^\varepsilon_{k+1}, \qquad w^\varepsilon_m(0) = 0$$

as long as $(a_0, v_0) := (a_0, \nabla\phi_0)$ and $(a_j, v_j) := (a_j, \nabla\phi_j)$ $(1\leqslant j\leqslant m)$ solve (1.6) and (5.1), respectively, where r^ε_{k+1} is $\varepsilon^{-k+1}(A^\varepsilon_0 - \sum_{j=0}^k \varepsilon^j A_j)$. If Assumption 1.8 is satisfied then r^ε_{k+1} is bounded in H^{s+1} as $\varepsilon\to 0$.

Proof of Proposition 5.7. By assumption, $(b^\varepsilon_{k_0}, w^\varepsilon_{k_0})$ is uniformly bounded in $L^\infty([0,T), H^{s-2k_0+1}\times H^{s-2k_0+2})$ in the limit $\varepsilon\to 0$. Therefore, extracting a subsequence, there exists a weak limit, denoted by (a_{k_0+1}, v_{k_0+1}), in the same class. Moreover, we obtain the bound

$$\sup_{t\in[0,T]}\left(\|a_{k_0+1}\|_{H^{s-2k_0+1}} + \|v_{k_0+1}\|_{H^{s-2k_0+2}}\right) \leqslant \eta_{k_0+1}(T).$$

by the lower semi-continuity of the weak limit. Since $(b^\varepsilon_{k_0}, w^\varepsilon_{k_0})$ solves (5.7)–(5.9), we see that (a_{k_0+1}, v_{k_0+1}) solves

$$(5.10)\quad \begin{cases}\partial_t a_j + \sum_{i_1+i_2=j} v_{i_1}\cdot\nabla a_{i_2} + \sum_{i_1+i_2=j}\frac{1}{2}a_{i_1}\nabla\cdot v_{i_2} - i\frac{1}{2}\Delta a_{j-1} = 0,\\ \partial_t v_j + \nabla\sum_{i_1+i_2=j}\frac{1}{2}v_i\cdot v_j + \lambda\nabla\sum_{i_1+i_2=j}(|x|^{-(n-2)} * \mathrm{Re}(a_{i_1}\overline{a_{i_2}})) = 0,\\ a_j(0) = A_j, \quad v_j(0) = 0.\end{cases}$$

for $j = k_0+1$. By the way, once we know (a_j, v_j) $(j = [0, k_0])$, we can solve this system directly by a standard argument and obtain unique solution (a_{k_0+1}, v_{k_0+1}) in the same space. Therefore, the above weak limit is the

unique solution to (5.10). We now define ϕ_{k_0+1} by

$$\phi_{k_0+1}(t) = -\int_0^t \left(\sum_{i_1+i_2=j} \frac{1}{2} v_i \cdot v_j + \lambda \sum_{i_1+i_2=j} (|x|^{-(n-2)} * \mathrm{Re}(a_{i_1}\overline{a_{i_2}})) \right) ds.$$

Then, $\nabla\phi_{k_0+1} = v_{k_0+1}$ holds by the uniqueness of (5.10). Hence, $\nabla\phi_{k_0+1}$ is the unique solution to (5.1) for $j = k_0+1$. Since $v_0 \in L^p$ and $v_j \in L^2$ $(j \geqslant 1)$,

$$\phi_{k_0+1} \in C([0,T]; Y^{s-2k_0+3}_{[\frac{2p}{2+p},\infty]\cap(1^{**},\infty],2}).$$

T is arbitrary, and so we obtain the proposition. □

Proof of Proposition 5.8. By assumption, we can define $(b^\varepsilon_{k_0+1}, w^\varepsilon_{k_0+1})$ solving (5.7)–(5.9). We will bound

$$\widetilde{E}_{k_0+1}(t) := \left\| b^\varepsilon_{k_0+1}(t) \right\|_{H^{s-2k_0-1}} + \left\| w^\varepsilon_{k_0+1}(t) \right\|_{H^{s-2k_0}}.$$

Recall that the quadratic part of (5.7)–(5.8) is the same of (1.5) up to a constant, and that the linear part of (5.7)–(5.8) is essentially the same form. Hence, mimicking the estimates in the proof of Theorem 4.6, we deduce that, for any fixed $T > 0$,

$$\frac{d}{dt}\widetilde{E}_{k_0+1}(t) \leqslant C_s(\varepsilon^{k_0+1}\widetilde{E}_{k_0+1}(t)^2 + \mu^\varepsilon_{k_0+1}\widetilde{E}_{k_0+1}(t) + c_{k_0+1}\nu^\varepsilon_{k_0+1}) \tag{5.11}$$

holds for all $t \in [0,T]$. Here, we have used two functions: First is

$$\mu^\varepsilon_{k_0+1} = \mu^\varepsilon_{k_0+1}(T) := \eta(T) + \sum_{j=1}^{k_0+1} \varepsilon^j \eta_j(T)$$

which bounds the linear part

$$\sup_{t\in[0,T]} \left(\left\| \sum_{\ell=0}^{k_0+1} \varepsilon^\ell a_\ell \right\|_{H^{s-2k_0+1}} + \|v_0\|_{Y^{s-2k_0+2}_{p,q}} + \left\| \sum_{\ell=1}^{k_0+1} \varepsilon^\ell v_\ell \right\|_{H^{s-2k_0}} \right)$$

and second is

$$\nu^\varepsilon_{k_0+1} = \nu^\varepsilon_{k_0+1}(T) := \eta_{k_0+1}(T) + \sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{i=\ell+1}^{k_0+1} \eta_i(T)\eta_{k_0+2+\ell-i}(T)$$

which is an upper bound of the constant terms

$$\sup_{t\in[0,T]} \Bigg(\frac{1}{2}\|\Delta a_{k_0}\|_{H^{s-2k_0-1}} + C\sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{\substack{1\leqslant i,j\leqslant k_0+1,\\ i+j=k_0+2+\ell}} \left(\|v_i\|_{H^{s-2k_0+2}} \|v_j\|_{H^{s-2k_0+2}} + \|a_i\|_{H^{s-2k_0+1}} \|a_j\|_{H^{s-2k_0+1}} \right) \Bigg)$$

up to an adjusting constant c_{k_0+1}. The constant C_s comes from (5.11). This constant is independent of k_0 because it has been already taken into account when we use $\mu^\varepsilon_{k_0+1}$ and $\nu^\varepsilon_{k_0+1}$.

We now show that $\sup_{t\in[0,T]} \widetilde{E}_{k_0+1}(t)$ is uniformly bounded for small ε. We keep fixing $T > 0$. By Assumption (1.8), we see that there exists a positive constant β_{k_0+1} depending only on A_0^ε such that $\widetilde{E}_{k_0+1}(0) \leqslant \beta_{k_0+1}$ holds for $\varepsilon \in (0,1]$. Set a function

$$Z_{k_0+1}(t) := \widetilde{E}_{k_0+1}(t)\exp(-C_s\mu^\varepsilon_{k_0+1}(T)t)$$

and two constants

$$\begin{aligned}\delta_{k_0+1} &:= (1+\sqrt{1+\beta_{k_0+1}})^{-1},\\ \theta_{m+1} &:= \frac{\delta\mu^\varepsilon_{k_0+1}(T)}{2c_{k_0+1}\nu^\varepsilon_{k_0+1}(T)(1-e^{-C_s\mu^\varepsilon_{k_0+1}(T)T})}.\end{aligned}$$

Then, multiplying the both sides of (5.11) by $\frac{\theta_{k_0+1}\exp(-C_st\mu^\varepsilon_{k_0+1})}{(1+\theta_{k_0+1}Z_{k_0+1}(t))^2}$, we obtain

$$\frac{\theta_{k_0+1}Z'_{k_0+1}(t)}{(1+\theta_{k_0+1}Z_{k_0+1}(t))^2} \leqslant C_s\varepsilon^{k_0+2}e^{C_st\mu^\varepsilon_{k_0+1}}\theta^{-1}_{k_0+1}+C_sc_{k_0+1}\nu^\varepsilon_{k_0+1}e^{-C_st\mu^\varepsilon_{k_0+1}}\theta_{k_0+1},$$

where we denote $\mu^\varepsilon_{k_0+1}(T)$ and $\nu^\varepsilon_{k_0+1}(T)$ by $\mu^\varepsilon_{k_0+1}$ and $\mu^\varepsilon_{k_0+1}$, respectively, for short. Integration over $[0,t]$ gives

$$\begin{aligned}\frac{1}{1+\theta_{k_0+1}Z_{k_0+1}(t)} \geqslant{}& \frac{1}{1+\theta_{k_0+1}\widetilde{E}_{k_0+1}(0)}\\ &-\frac{c_{k_0+1}\nu^\varepsilon_{k_0+1}}{\mu^\varepsilon_{k_0+1}}(1-e^{-C_st\mu^\varepsilon_{k_0+1}})\theta_{k_0+1}-\frac{\varepsilon^{k_0+2}}{\mu^\varepsilon_{k_0+1}}(e^{C_st\mu^\varepsilon_{k_0+1}}-1)\theta^{-1}_{k_0+1}.\end{aligned} \tag{5.12}$$

Let us show that the right hand side of (5.12) is bounded by $\delta_{k_0+1}/2$ from below. For simplicity, in the following, we omit the index k_0+1 and denote β_{k_0+1}, c_{k_0+1}, δ_{k_0+1}, $\mu^\varepsilon_{k_0+1}$, $\nu^\varepsilon_{k_0+1}$, and θ_{k_0+1} by β, c, δ, μ^ε, ν^ε, and θ, respectively. We also omit T variable in $\eta(T)$ and $\eta_j(T)$. By the fact that $\eta_{j+1} \gg \eta_j$ for each j and large T and by definitions of μ^ε and ν^ε, $c\nu^\varepsilon \geqslant \mu^\varepsilon$ holds for all $\varepsilon \in [0,1]$ if T is large. Then, replacing T with larger one if necessary, we obtain

$$\begin{aligned}\frac{1}{1+\theta\widetilde{E}_{k_0+1}(0)}-\delta \geqslant{}& \frac{1}{1+\theta\beta}-\delta = \frac{e^{C_s\mu^\varepsilon T}-1}{e^{C_s\mu^\varepsilon T}(1+\frac{\mu^\varepsilon}{c\nu^\varepsilon}\frac{\delta\beta}{2})-1}-\delta\\ &\geqslant \frac{e^{C_s\mu^\varepsilon T}(2-2\delta-\delta^2\beta)-2+2\delta}{e^{C_s\mu^\varepsilon T}(2+\delta\beta)-2} \geqslant \frac{2-2\delta-\delta^2\beta}{2(2+\delta\beta)} = \frac{\delta}{2},\end{aligned} \tag{5.13}$$

where we have used the relation $1-2\delta-\delta^2\beta = 0$. Moreover, θ is the minimizer of the quantity

$$\frac{c\nu^\varepsilon}{\mu^\varepsilon}(1-e^{-C_s\mu^\varepsilon T})\theta^2-\delta\theta+\frac{\varepsilon^{k_0+2}}{\mu^\varepsilon}(e^{C_s\mu^\varepsilon T}-1)$$

and so this quantity becomes less than or equal to zero if

$$\varepsilon \leqslant \left(\frac{\delta^2(\mu^\varepsilon)^2e^{C_s\mu^\varepsilon T}}{c\nu^\varepsilon(e^{C_s\mu^\varepsilon T}-1)^2}\right)^{\frac{1}{k_0+2}}. \tag{5.14}$$

We now replace this condition with stronger but clearer one. We first let ε be so small that

$$\varepsilon \leqslant \min_{j\in[1,k_0+1]} \left(\frac{\eta}{\eta_j}\right)^{\frac{1}{j}} = \min_{j\in[1,k_0+1]} \frac{\eta}{\Gamma_j^{1/j} e^{(2-1/j)C_s\eta(T)T}}. \tag{5.15}$$

For such ε, we have $\mu^\varepsilon \leqslant (k_0+2)\eta$ and, by definition of η_j (5.6),

$$\begin{aligned}
\nu^\varepsilon &= \eta_{k_0+1} + \sum_{\ell=0}^{k_0} \varepsilon^\ell \sum_{i=\ell+1}^{k_0+1} \eta_i \eta_{k_0+2+\ell-i} \\
&\leqslant \eta_{k_0+1} + \widetilde{\Gamma}_1 \frac{e^{(2k_0+2)C_s\eta T}}{\eta^{k_0}} + \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0+1}} \sum_{\ell=1}^{k_0} \widetilde{\Gamma}_2 \eta \left(\frac{\varepsilon}{\eta} e^{(2-1/\ell)C_s\eta T}\right)^\ell \\
&\leqslant \eta_{k_0+1} + \widetilde{\Gamma}_1 \frac{e^{(2k_0+2)C_s\eta T}}{\eta^{k_0}} + \widetilde{\Gamma}_3 \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0}} \leqslant \widetilde{\Gamma}_4 \frac{e^{(2k_0+3)C_s\eta T}}{\eta^{k_0}},
\end{aligned}$$

where $\widetilde{\Gamma}_i$ is a constant depending on k_0 and Γ_j $(1 \leqslant j \leqslant k_0+1)$. Therefore, the right hand side of (5.14) is bounded below by

$$\begin{aligned}
\left(\frac{\delta^2(\mu^\varepsilon)^2 e^{C_s\mu^\varepsilon T}}{c\nu^\varepsilon (e^{C_s\mu^\varepsilon T}-1)^2}\right)^{\frac{1}{k_0+2}} &\geqslant \left(\frac{\delta^2\eta^2}{c\nu^\varepsilon e^{C_s\mu^\varepsilon T}}\right)^{\frac{1}{k_0+2}} \\
&\geqslant \widetilde{\Gamma}_5 \left(\frac{\eta^2}{(\eta^{-k_0} e^{(2k_0+3)C_s\eta T}) e^{(k_0+2)C_s\eta T}}\right)^{\frac{1}{k_0+2}} \\
&= \widetilde{\Gamma}_5 \frac{\eta}{e^{(3-\frac{1}{k_0+2})C_s\eta T}} \geqslant \widetilde{\Gamma}_5 \frac{\eta}{e^{3C_s\eta T}} =: \varepsilon^{k_0+2},
\end{aligned}$$

where $\widetilde{\Gamma}_5$ depends on $\widetilde{\Gamma}_4$, β, and c. Then, the condition $\varepsilon \leqslant \varepsilon_{k_0+2}$ ensures (5.14) and so

$$\delta - \frac{c\nu^\varepsilon}{\mu^\varepsilon}(1-e^{-C_s\mu^\varepsilon T})\theta - \frac{\varepsilon^{k_0+2}}{\mu^\varepsilon}(e^{C_sT\mu^\varepsilon}-1)\theta^{-1} \geqslant 0. \tag{5.16}$$

Note that ε_{k_0+2} is smaller than the right hand side of (5.15) and so that $\varepsilon \leqslant \varepsilon_{k_0+2}$ is stronger than (5.15).

Furthermore, plugging (5.13) and (5.16) to (5.12), we obtain

$$\sup_{t\in[0,T]} \widetilde{E}_{k_0+1}(t) \leqslant 3\sqrt{1+\beta}\,\theta^{-1} e^{C_s\mu^\varepsilon T} \leqslant \frac{6c\sqrt{1+\beta}\,\nu^\varepsilon}{\delta\mu^\varepsilon} e^{C_s\mu^\varepsilon T}, \tag{5.17}$$

which is the desired bound. Indeed, the right hand side is bounded by

$$\frac{6c\sqrt{1+\beta}\,\widetilde{\Gamma}_4}{\delta\eta^{k_0+1}} e^{C_s(3k_0+5)\eta T}$$

as long as $\varepsilon \leqslant \varepsilon_{k_0+2}$. We finally confirm that the right hand side of (5.17) tends to $\eta_{k_0+2}(T)$ defined by (5.6) with a suitable constant. It holds that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mu^\varepsilon = \lim_{\varepsilon\to 0} \mu^\varepsilon_{k_0+1}(T) = \eta(T), \\
&\lim_{\varepsilon\to 0} \nu^\varepsilon = \lim_{\varepsilon\to 0} \nu^\varepsilon_{k_0+1}(T) = \eta_{k_0+1} + \sum_{i=1}^{k_0+1} \eta_i \eta_{k_0+2-i} \leqslant \frac{\hat{\Gamma}_{k_0+2}}{\eta(T)^{k_0}} e^{(2k_0+2)C_s\eta(T)T},
\end{aligned}$$

where $\hat{\Gamma}_{k_0+2}$ depends on k_0 and Γ_j $(1 \leqslant j \leqslant k_0+1)$. Therefore, we end up with the estimate

$$\begin{aligned}\limsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \widetilde{E}_{k_0+1}(t) &\leqslant \frac{6\sqrt{1+\beta}(\hat{\Gamma}_{k_0+2}\eta(T)^{-k_0}e^{(2k_0+2)C_s g(T)T})}{\delta\eta(T)}e^{C_s\eta(T)T}\\ &=: \frac{\Gamma_{k_0+2}}{\eta(T)^{k_0+1}}e^{(2k_0+3)C_s\eta(T)T} = \eta_{k_0+2}(T),\end{aligned}$$

which completes the proof. □

Appendix A. Global existence of solution to the Euler-Poisson equations

In this section we give the proof of Theorem 2.1. Let us consider

$$\text{(A.1)}\qquad \begin{cases} \rho_t + r^{-(n-1)}\partial_r(r^{n-1}\rho v) = 0, & \rho(0,r) = \rho_0(r);\\ v_t + v\partial_r v + \lambda\partial_r V_{\mathrm{P}} = 0, & v(0,r) = v_0(r);\\ -r^{-(n-1)}\partial_r(r^{n-1}V_{\mathrm{P}}) = \rho,\ V_{\mathrm{P}}\in L^\infty, & V_{\mathrm{P}}\to 0 \text{ as } r\to\infty \end{cases}$$

Here, $n \geqslant 1$ denotes space dimensions, $r \geqslant 0$ denotes the distance from the origin, and λ is a given physical constant. Unknowns are the mass density $\rho = \rho(t,r) \geqslant 0$ and the velocity field $v = v(t,r) \in \mathbb{R}$. If $n = 1$ or 2, we change the condition for Poisson equation of (A.1) into $V_{\mathrm{P}}(t,0) = 0$, $\partial_r V_{\mathrm{P}} \in L^\infty$, and $\partial_r V_{\mathrm{P}} \to 0$ as $r \to \infty$. In other words, we let V_{P} be as

$$\text{(A.2)}\qquad V_{\mathrm{P}}(t,r) = \begin{cases} \displaystyle\int_0^r s^{-(n-1)}\int_0^s \rho(t,\sigma)\sigma^{n-1}d\sigma ds & \text{if } n = 1,2,\\ \displaystyle\int_r^\infty s^{-(n-1)}\int_0^s \rho(t,\sigma)\sigma^{n-1}d\sigma ds & \text{if } n \geqslant 3. \end{cases}$$

This is well-defined because we restrict our attention to ρ belonging to $C([0,\infty)) \cap L^1((0,\infty), r^{n-1}dr)$. One can verify that the condition $V_{\mathrm{P}}(t,r) \to 0$ as $r \to \infty$ is not suitable for $n = 1$ or 2. Remark that (A.1) is a radial version of the compressible Euler-Poisson equations

$$\text{(A.3)}\qquad \begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0, & \rho(0,x) = \rho_0(x),\\ \partial_t v + (v\cdot\nabla)v + \lambda\nabla V_{\mathrm{P}} = 0, & v(0,x) = v_0,\\ -\Delta V_{\mathrm{P}} = \rho, \quad V_{\mathrm{P}} \in L^\infty(\mathbb{R}^n), & V_{\mathrm{P}} \to 0 \text{ as } |x| \to \infty. \end{cases}$$

A.1. **Reduction to an ODE for characteristic curves.** We follow the argument in [12, 21]. Define characteristic curve X as a function $\mathbb{R}_+ \to \mathbb{R}_+$ with parameter $R \in \mathbb{R}_+$ which is defined by an ODE

$$\frac{d}{dt}X(t,R) = v(t,X(t,R)), \quad X(0,R) = R.$$

Let $m(t,r) := \int_0^r \rho(t,s)s^{n-1}ds$. In the followings, we denote $\partial_t X$ as X' by the respect that R is a parameter. Then, (A.1) is reduced to the following ODE for characteristic curve

$$\text{(A.4)}\qquad X''(t,R) = -\frac{\lambda m_0(R)}{X(t,R)^{n-1}}, \quad X'(0,R) = v_0(R), \quad X(0,R) = R,$$

where $m_0(r) := \int_0^r \rho_0(s)s^{n-1}ds$. This reduction is the same spirit as the use of the Lagrangian coordinate (see [23, 24]). Put

$$B(t,R) := \exp\left(\int_0^t \partial_r u(\tau, X(\tau,R))d\tau\right).$$

It holds that $B(t,R) = \partial_R X(t,R)$. The solution to (A.1) is given explicitly in terms of X and B as

$$\rho(t,X(t,R)) = \frac{R^{n-1}\rho_0(R)}{X^{n-1}B(t,R)}, \qquad v(t,X(t,R)) = \frac{d}{dt}X(t,R). \tag{A.5}$$

As in [21], we introduce the quantity

$$C(r) := v_0^2(r) + \frac{2\lambda m_0(r)}{(n-2)r^{n-2}}$$

for $n \geqslant 3$. It can be said that this describes the balance between the initial velocity and the strength of the force governed by the Poisson equation. This clarifies the description of the conditions for global existence. The large time behavior of X is also distinguished by C (Remark A.3). For the proof of Theorem 2.1, we use two propositions (Propositions A.1 and A.2, below). We first prove Proposition A.1 and then prove the theorem.

A.2. **The necessary and sufficient condition for the attractive case.** We first consider the case $\lambda < 0$. We use the function space D^k defined in (2.3). The following result is announced but not proven in [21].

Proposition A.1 (Critical thresholds for $\lambda < 0$ case)**.** *Suppose $\lambda < 0$, $n \geqslant 1$, $\rho_0 \in D^k$, and $v_0 \in D^{k+1}$ with $v_0(0) = 0$ for an integer $k \geqslant 0$.*

(1) *If $n = 1$ or 2 then the solution to* (A.1) *is global if and only if $\rho_0(r) = 0$, $v_0(r) \geqslant 0$, and $v_0'(r) \geqslant 0$ hold for all $r \geqslant 0$. In particular, if $\rho_0 \not\equiv 0$ then the solution breaks down in finite time.*
(2) *If $n \geqslant 3$ then the solution is global if and only if $v_0(r) \geqslant 0$, $C(r) \geqslant 0$, and $C'(r) \geqslant 0$ hold for all $r \geqslant 0$.*

If ρ_0 and v_0 satisfy the condition for global existence, then the corresponding solution of (A.1) *satisfies*

$$\begin{aligned}&\rho \in C^2([0,\infty), D^m) \cap C^\infty((0,\infty), D^m),\\ &v \in C^1([0,\infty), D^{m+1}) \cap C^\infty((0,\infty), D^{m+1}).\end{aligned}$$

The solution is unique in $C^2([0,\infty), D^0) \times C^2([0,\infty), D^1)$ and also solves (A.3) *in the distribution sense.*

Proof. Let us recall some facts from [21]. Under the assumptions of Proposition A.1, we deduce from Proposition 2.4 in [21] that (A.1) has a unique solution

$$\begin{aligned}&\rho \in C^2([0,T), D^k) \cap C^\infty((0,T), D^k),\\ &v \in C^1([0,T), D^{k+1}) \cap C^\infty((0,T), D^{k+1}),\end{aligned}$$

provided there exists T such that, for $t \in [0,T]$, $X(t,R) > 0$, $R > 0$ and $B(t,R) > 0$, $R \geqslant 0$. Moreover, if $B(t_c, R_c) = 0$ holds for some (t_c, R_c) then the solution breaks down at $t = t_c$ (see, Corollary 5.2 in [12] or Proposition

2.3 in [21]). Furthermore, if $X(t_0, R_0) = 0$ for some (t_0, R_0) then such (t_c, R_c) exists and $t_c \leqslant t_0$, $R_c \leqslant R_0$ (see, Lemma 2.9 in [21]).

Step 1. We begin with the one-dimensional case. If ρ_0 is not identically zero, then we can choose R_0 so that $m_0(R_0) > 0$. Twice integration of (A.4) yields $X(t, R_0) = R_0 + v_0(R_0)t - (|\lambda| m_0(R_0)/2)t^2$. Therefore, we can find t_0 such that $X(t_0, R_0) = 0$, which leads to the finite-time breakdown of the solution. On the other hand, if $\rho_0 \equiv 0$ then $X(t, R) = R + v_0(R)t$ and $B(t, R) = 1 + v_0'(R)t$. Hence, the solution is global if and only if $v_0(R) \geqslant 0$ and $v_0'(R) \geqslant 0$ holds for all $R \geqslant 0$.

Step 2. We next treat the two-dimensional case. If there exists R_0 such that $m_0(R_0) > 0$. Multiplying (A.4) by X', we obtain

$$0 \leqslant (X'(t, R_0))^2 = v_0(R_0)^2 - 2|\lambda| m_0(R_0) \log \left(\frac{X(t, R_0)}{R_0} \right),$$

which yields an upper bound of X:

$$X(t, R_0) \leqslant R_0 \exp \left(\frac{v_0(R_0)^2}{2|\lambda| m_0(R_0)} \right) =: X_{\mathrm{ub}}.$$

Plugging this to (A.4), we see that

$$X''(t, R_0) \leqslant -\frac{|\lambda| m_0(R_0)}{X_{\mathrm{ub}}} < 0.$$

Therefore, there exists t_0 such that $X(t_0, R_0) = 0$. In the case where $\rho \equiv 0$, by the same argument as in the one-dimensional case, we see that the solution is global if and only if $v_0(R) \geqslant 0$ and $v_0'(R) \geqslant 0$ hold for all $R \geqslant 0$.

Step 3. Let us proceed to $n \geqslant 3$ case. For simplicity, we use

$$A(r) := \frac{2|\lambda| m_0(r)}{n-2} \geqslant 0.$$

Notice that C is written as $C(r) = v_0(r)^2 - A(r)/r^{n-2}$. We first note that $v_0 \geqslant 0$ is necessary for global existence. Indeed, if $v_0(R_0) < 0$ for some $R_0 > 0$, then $X''(t, R_0) \leqslant 0$ follows from (A.4) and so $X'(t, R) \leqslant X'(0, R) = v_0(R) < 0$. Hence, there exists t_0 such that $X(t_0, R_0) = 0$. We next show that $C \geqslant 0$ is also necessary for global existence. Assume that there exists R_0 such that $C(R_0) < 0$. In this case, $A(R_0) > 0$ by definition of C. Then, multiplying (A.4) by X', we obtain

$$0 \leqslant (X'(t, R_0))^2 = C(R_0) + \frac{A(R_0)}{X(t, R_0)^{n-2}}.$$

This yields an upper bound of X:

$$X(t, R_0) \leqslant \left(\frac{|C(R_0)|}{A(R_0)} \right)^{\frac{1}{n-2}}.$$

Then, the same argument as in the two-dimensional case shows the existence of t_0 such that $X(t_0, R_0) = 0$. Therefore, $C \geqslant 0$ is necessary for global existence.

In the following, we suppose $v_0 \geqslant 0$ and $C \geqslant 0$ are satisfied. Under this restriction, let us show that the solution is global if and only if $C'(R) \geqslant 0$ holds for all $R \geqslant 0$. What to show is that

$$C'(R) \geqslant 0 \Longleftrightarrow B(t, R) > 0, \quad \forall t \geqslant 0. \tag{A.6}$$

We first consider the case $v_0(R) > 0$. Then, $C(R) > 0$ or $A(R) > 0$ hold. Moreover, $X(t,R) \to \infty$ as $t \to \infty$ since $X''(t,R) \geqslant 0$ and so $X'(t,R) \geqslant X'(0,R) = v_0(R) > 0$. In this case, by multiplication of (A.4) with X',

$$X'(t,R) = \sqrt{C(R) + \frac{A(R)}{X(t,R)^{n-2}}} > 0,$$

and so

$$\int_R^{X(t,R)} \frac{dy}{\sqrt{C(R) + A(R)y^{-(n-2)}}} = t.$$

Differentiate with respect to R to obtain

$$\frac{B(t,R)}{X'(t,R)} - \frac{1}{v_0(R)} - \frac{1}{2}\int_R^{X(t,R)} \frac{C'(R) + A'(R)y^{-(n-2)}}{\big(C(R) + A(R)y^{-(n-2)}\big)^{3/2}} dy = 0.$$

We put

$$\widetilde{B}(t,R) := \frac{B(t,R)}{X'(t,R)} = \frac{1}{v_0(R)} + \frac{1}{2}\int_R^{X(t,R)} \frac{C'(R) + A'(R)y^{-(n-2)}}{\big(C(R) + A(R)y^{-(n-2)}\big)^{3/2}} dy.$$

Two quantity $\widetilde{B}$ and B have the same sign. Notice that

$$A'(R) = \frac{2\lambda}{n-2}\rho_0(R)R^{n-1} \geqslant 0$$

and that the denominator in the last integral is always positive. Therefore, if $C'(R) \geqslant 0$ then the above integral is nonnegative, and so $\widetilde{B}(t,R)$ stays positive for all $t \geqslant 0$. On the other hand, if $C'(R) < 0$ then the integral in $\widetilde{B}(t,R)$ tends to $-\infty$ as $t \to \infty$. This is because, choosing X_0 so large that $C'(R) + A'(R)X_0^{-(n-2)} < -|C'(R)|/2$, we have

$$\int_{X_0}^{X(t,R)} \frac{C'(R) + A'(R)y^{-(n-2)}}{\big(C(R) + A(R)y^{-(n-2)}\big)^{3/2}} dy < -\int_{X_0}^{X(t,R)} \frac{|C'(R)|y^{3(n-2)/2}}{2A(R)^{3/2}} dy$$

if $A(R) > 0$ and

$$\int_{X_0}^{X(t,R)} \frac{C'(R) + A'(R)y^{-(n-2)}}{\big(C(R) + A(R)y^{-(n-2)}\big)^{3/2}} dy < -\int_{X_0}^{X(t,R)} \frac{|C'(R)|}{2C(R)^{3/2}} dy$$

if $C(R) > 0$. The right hand sides of both inequalities tend to $-\infty$ as $t \to \infty$. Therefore, we can choose t_c such that $B(t_c, R) = 0$.

We finally discuss the case where $v_0(R) = 0$. In this case, since $C(R) \geqslant 0$, we have $C(R) = 0$ and so $A(R) = 0$ ($m_0(R) = 0$) by the definition of C. It implies that $\rho(r) = 0$ for all $r \leqslant R$ and so that, for all $r \leqslant R$, $X'(t,r) \equiv 0$ and $X(t,r) \equiv r$. Hence, by continuity of B, one verifies that $B(t,R) = \lim_{r\uparrow R} \partial_R X(t,r) = 1 > 0$ for all $t \geqslant 0$. Note that $C'(R) = 0$ since $\rho(R) = 0$. Thus, (A.6) is justified. $\square$

A.3. **The necessary condition for the repulsive case.** The following result is Remark 5.4 of [12] if we restrict our attention to the case where $v_0 \geqslant 0$, and this is also a part of Theorem 1.7 in [21].

Proposition A.2 (Necessary condition for $\lambda > 0$ case)**.** *Let $\lambda > 0$, $n \geqslant 3$, $\rho_0 \in D^0$, and $v_0 \in D^1$ with $v_0(0) = 0$. Then, the classical solution of* (A.1) *is global only if $C'(R) \geqslant 0$ for all $R \geqslant 0$.*

A.4. **Proof of Theorem 2.1.**

Proof of Theorem 2.1. By Proposition A.1, the solution breaks down in finite time if $n = 1$, 2 and $\lambda < 0$, since $\rho_0 \not\equiv 0$. Suppose $n \geqslant 3$. By assumptions on the initial data, we have $C(0) = 0$ and $C(r) \to 0$ as $r \to \infty$. Now, Propositions A.1 and A.2 imply that the solution is global only if $C'(R) \geqslant 0$ for all $R \geqslant 0$, that is, only if $C \equiv 0$. In the $\lambda < 0$ case, Proposition A.1 shows the solution is global if we take the positive root:

$$v_0(R) = \sqrt{\frac{A(R)}{R^{n-2}}} \geqslant 0. \tag{A.7}$$

If $\lambda > 0$ then $C \equiv 0$ implies $\rho \equiv 0$, which is excluded by assumption.

If $n \geqslant 3$, $\lambda < 0$, and $v_0(R)$ is given as (A.7) then $C \equiv 0$ and so X satisfies the equation

$$X'(t,R) = \sqrt{\frac{A(R)}{X(t,R)^{n-2}}}, \quad X(0,R) = R.$$

By separation of variables, we obtain $X(t,R) = R(1 + \frac{nv_0(R)}{2R}t)^{2/n}$. Then, (A.5) gives the explicit representation of the solution. □

Remark A.3. The value $C(R)$ is useful to describe the large time behavior of $X(t,R)$ for $n \geqslant 3$. In previous results, we have already established the estimate

$$C_1 t^{2/n} + o(t^{2/n}) \leqslant X(t,R) \leqslant C_2 t + o(t)$$

for a constant C_1 and $C_2 = \sqrt{C(R)}$ in some cases (see [12, Remark 5.1]). Notice that the lower bound is $O(t^{2/n})$ and the upper bound is $O(t)$ as $t \to \infty$. This estimate is sharp in such a sense that, as $t \to \infty$, the both cases $X(t,R) = O(t^{2/n})$ and $X(t,R) = O(t)$ can happen. Now, we summarize as follows: Let $n \geqslant 3$, $\lambda \in \mathbb{R} \setminus \{0\}$, $v_0(R) \in \mathbb{R}$, $m_0(R) > 0$, and

$$C(R) = v_0(R)^2 + \frac{2\lambda m_0(R)}{(n-2)R^{n-2}}.$$

Let $X(t,R)$ be a solution of (A.4).

- If $C(R) > 0$, then $X(t,R) > 0$ for all $t \geqslant 0$ and $X(t,R) = \sqrt{C(R)}t + o(t)$ as $t \to \infty$.
- If $C(R) = 0$, then $\lambda < 0$ and
$$X(t,R) = R\left(1 + t\sqrt{\frac{|\lambda| n^2 m_0(R)}{2(n-2)R^n}}\right)^{\frac{2}{n}} = O(t^{\frac{2}{n}})$$
as $t \to \infty$.
- If $C(R) < 0$, then $\lambda < 0$ and there exists $t_c < \infty$ such that $X(t_c, R) = 0$.

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